A PRACTICAL INTRODUCTION TO THE GSPRT

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1. DESCRIPTION OF THE GSPRT

Let \( p(\theta, x) \) be a parametrized distribution and let \( L(\theta, x) = \sum_{i=1}^{N} \log p(\theta, x_i) \) be the corresponding log-likelihood function for \( N \) independent trials.

Let \( \phi(\theta) \) be a function of the parameters and assume given \( \phi_0, \phi_1 \in \mathbb{R} \). The Generalized Sequential Probability Likelihood Ratio Test [1] for \( \phi = \phi_0 \) versus \( \phi = \phi_1 \) is based on monitoring the difference

\[
\text{LLR} := L(\hat{\theta}_1, x) - L(\hat{\theta}_0, x)
\]

where \( \hat{\theta}_i \) is the maximum likelihood estimator for \( \theta \) subject to the condition \( \phi(\hat{\theta}_i) = \phi_i \). I.e.

\[
\hat{\theta}_i = \arg \max_{\phi(\theta) = \phi_i} L(\theta, x)
\]

Like for the SPRT the test stops when LLR leaves the interval \([\log(\beta/(1-\alpha)), \log((1-\beta)/\alpha)]\) where as usual \( \alpha, \beta \) stand for the Type I,II error probabilities.

2. AN APPROXIMATION

Let \( \hat{\theta} \) be the unconstrained maximum likelihood estimator for \( \theta \), i.e.

\[
\hat{\theta} = \arg \max_{\theta} L(\theta, x)
\]

and put \( \hat{\phi} = \phi(\hat{\theta}) \). Let \( V(\hat{\phi}) \) be an estimator for the variance \( \text{Var}(\hat{\phi}) \) of \( \hat{\phi} \) with relative error \( O(1/\sqrt{N}) \) (thus any standard estimator will do). We claim that under suitable regularity conditions we have the following very convenient approximation for (1.1)

\[
L(\hat{\theta}_1, x) - L(\hat{\theta}_0, x) \approx \frac{1}{2} \frac{(\phi_1 - \phi_0)(2\hat{\phi} - \phi_0 - \phi_1)}{V(\phi)}
\]

This means that the, sometimes quite cumbersome, calculation of the conditional estimates \( \hat{\theta}_0, \hat{\theta}_1 \) is not actually required.

Example. Assume we take independent trials from a multinomial distribution with probabilities \((p_i)_{i=1,...,n}\). Let \( \phi = \sum_{i=1}^{n} a_i p_i \) for given \((a_i)_{i=1,...,n} \in \mathbb{R} \). Assume that after \( N \) trials the outcome frequencies are \((N_i)_{i=1,...,n} \) (with \( N = \sum_{i=1}^{n} N_i \)). Then to calculate (2.1) we calculate first the empiric probabilities \( \hat{p}_i := N_i/N \) and then
we put
\[ \hat{\phi} = \sum_{i=1}^{n} a_i \hat{p}_i \]
\[ V(\hat{\phi}) = \frac{1}{N} (-\hat{\phi}^2 + \sum_i a_i^2 \hat{p}_i) \]

This example is relevant for chess engine testing [2] in which case \( \phi \) would stand for the expected score of a match. For the naive trinomial model one takes \((p_1, p_2, p_3) = (w, d, l)\) and \(a_1 = 1, a_2 = 1/2, a_3 = 0\) whereas in the 5-nomial model (for paired games with reversed colors) one takes \((p_1, p_2, p_3, p_4, p_5) = (p_2, p_3/2, p_1, p_1/2, p_0)\) with \(a_1 = 1, a_2 = 3/4, a_3 = 1/2, a_4 = 1/4, a_5 = 0\). Note that in the 5-nomial model \(N\) is the number of games divided by two (one trial consists of two games).

3. Derivation of the approximation

For simplicity we will give the derivation for one particular choice of \(V(\hat{\phi})\). One may check that the relative change in the right hand side of (2.1), when replacing one \(V(\hat{\phi})\) by another, goes to zero when \(N\) goes to infinity.

In order to verify (2.1) the first mission is to calculate \(\hat{\theta}_i\). Using Lagrange multipliers we see that we have to solve \((i \in \{0, 1\})\)
\[ \nabla_{\hat{\theta}} L(\hat{\theta}_i, x) = \lambda \nabla_{\hat{\theta}} \phi(\hat{\theta}_i) \]
\[ \phi(\hat{\theta}_i) = \phi_i \]

If \(\hat{\phi} = \phi_i\) then \(\lambda = 0, \hat{\theta}_i = \hat{\theta}\). We will assume that \(\hat{\phi}\) is close to \(\phi_i\) so that \(\lambda\) is small. We get in first order (\(H = \text{Hessian}\))
\[ (H_{\hat{\theta}} L(\hat{\theta}_i, x) - \lambda H_{\hat{\theta}} \phi(\hat{\theta}_i)) \cdot (\hat{\theta}_i - \hat{\theta}) = \lambda \nabla_{\hat{\theta}} \phi(\hat{\theta}_i) \]
\[ \nabla_{\hat{\theta}} \phi(\hat{\theta}_i)^t \cdot (\hat{\theta}_i - \hat{\theta}) = \phi_i - \hat{\phi} \]

and hence
\[ \hat{\theta}_i - \hat{\theta} = \lambda H_{\hat{\theta}} L(\hat{\theta}_i, x)^{-1} \cdot \nabla_{\hat{\theta}} \phi(\hat{\theta}_i) \]
\[ \lambda \nabla_{\hat{\theta}} \phi(\hat{\theta}_i)^t \cdot H_{\hat{\theta}} L(\hat{\theta}_i, x)^{-1} \cdot \nabla_{\hat{\theta}} \phi(\hat{\theta}_i) = \phi_i - \hat{\phi} \]

Write \(V(\hat{\phi}) = -\nabla_{\hat{\theta}} \phi(\hat{\theta}_i)^t \cdot H_{\hat{\theta}} L(\hat{\theta}_i, x)^{-1} \cdot \nabla_{\hat{\theta}} \phi(\hat{\theta}_i)\). It is well-known that \(V(\hat{\phi})\) is an approximation for the variance \(\text{Var}(\hat{\phi})\) of \(\hat{\phi}\). Then we get by eliminating \(\lambda\) from (3.1)
\[ \hat{\theta}_i - \hat{\theta} = -\frac{\phi_i - \hat{\phi}}{V(\hat{\phi})} H_{\hat{\theta}} L(\hat{\theta}_i, x)^{-1} \cdot \nabla_{\hat{\theta}} \phi(\hat{\theta}_i) \]

Now we have (using the fact that \(\hat{\theta}\) is extremal for \(L(\hat{\theta}, x)\))
\[ L(\hat{\theta}, x) \cong L(\hat{\theta}_i, x) + \frac{1}{2} (\hat{\theta}_i - \hat{\theta})^t \cdot H_{\hat{\theta}} L(\hat{\theta}_i, x) \cdot (\hat{\theta}_i - \hat{\theta}) \]
\[ \cong L(\hat{\theta}_i, x) - \frac{1}{2} (\phi_i - \hat{\phi})^2 V(\hat{\phi}) \]

Substituting this in (1.1) yields what we want.
REFERENCES
