NOTES ON CONTINUOUS RANDOM WALKS
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The formula which is implemented in the Fishtest Framework is (6.1). It is basically [1, Corollary 3.44]. However I first derived the formula myself before I discoved this reference and I was too lazy afterwards to do the translation.

1. Continuous random walks

We discuss a continuous 1-dimensional random walk with drift μ and variance σ^2 per time unit. We assume there is some boundary C in the $x-t$-plane (t is the time coordinate and x is a spatial coordinate, by convention we assume that t is horizontal) such that if the random walk touches C there is a payoff of $\psi(x,t)$.

Let $P(x,t)$ be the expectation value of the eventual payoff for a particle at (x,t) Then the value of $P(x,t)$ is governed by the following diffusion equation

$$\frac{\partial P}{\partial t} = -\mu \frac{\partial P}{\partial x} - \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial x^2} \tag{1.1}$$

2. Stopping time

The stopping time S is defined as the first time that the random walk touches the boundary of an interval of length A. Below we compute the probability density of S.

To do this we first compute the probability that S is larger than a given constant T. I.e. $P(S \geq T)$. To do this have to solve (1.1) with the boundary conditions $P(0,t) = P(A,t) = 0$ for $t < T$ and $P(x,T) = 1$. We will look at the more general problem where $P(x,T) = \psi(x)$.

Using separation of variables $P(x,t) = X(x)T(t)$ we have to solve

$$\frac{\partial T}{\partial t} = \lambda T$$

$$\frac{\sigma^2}{2} \frac{\partial^2 X}{\partial x^2} + \mu \frac{\partial X}{\partial x} + \lambda X = 0$$

with $X(0) = 0$, $X(A) = 0$.

The roots of the characteristic equation of the last equation are

$$\chi_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 2\sigma^2 \lambda}}{\sigma^2}$$

It follows that we should have

$$\Delta = 2\sigma^2 \lambda - \mu^2 > 0$$

and the general solutions are

$$e^{-x\mu/\sigma^2} \sin x\sqrt{\Delta}/\sigma^2$$

as the cosine solutions are excluded by our boundary condition $X(0) = 0$. The boundary condition $X(A) = 0$ leads to

$$A\sqrt{\Delta}/\sigma^2 = n\pi$$

which leads to the following values for λ:

$$\lambda_n = \frac{n^2\pi^2\sigma^2}{2A^2} + \frac{\mu^2}{2\sigma^2}$$

Hence our general solution is now

$$e^{\lambda_n t - x\mu/\sigma^2} \sin n\pi x/A$$

We look for a_n such that

$$\sum_n a_n e^{\lambda_n T - x\mu/\sigma^2} \sin n\pi x/A = \psi(x)$$

or equivalently b_n such that

$$\sum_n b_n \sin n\pi x/A = e^{\gamma x}\psi(x)$$

where $\gamma = \mu/\sigma^2$ and $b_n = a_n e^{\lambda_n T}$. Standard Fourier analysis yields

$$b_n = \frac{2}{A}\int_0^A e^{\gamma y} \sin \frac{n\pi y}{A} \psi(y)\, dy$$

So the solution is

$$P(x,t) = \sum_n e^{-\lambda_n T} e^{\lambda_n t - \gamma x} \sin \frac{n\pi x}{A} \left[\frac{2}{A}\int_0^A e^{\gamma y} \sin \frac{n\pi y}{A} \psi(y)\, dy\right] \tag{2.1}$$

Our original problem corresponds to

$$\psi(x) = \begin{cases} 1 & 0 \le x \le A \\ 0 & \text{otherwise} \end{cases}$$

Thus the probability that the stopping time S is $\ge T$ is

$$P(S \ge T) = \sum_n e^{-\lambda_n T} e^{-\gamma x} \sin \frac{n\pi x}{A} \left[\frac{2}{A}\int_0^A e^{\gamma y} \sin \frac{n\pi y}{A}\, dy\right]$$

For the sequel it will be convenient to define

$$\begin{aligned} H_n(\gamma, x) &= \frac{2}{A}\int e^{\gamma x} \sin \frac{n\pi x}{A}\, dx \\ &= \frac{2A\gamma e^{\gamma x} \sin \pi n x/A - 2\pi n e^{\gamma x} \cos \pi n x/A}{A^2\gamma^2 + \pi^2 n^2} \end{aligned}$$

In particular we have the following special values

$$H_n(\gamma, 0) = -\frac{2\pi n}{A^2\gamma^2 + \pi^2 n^2}$$

$$H_n(\gamma, A) = -(-1)^n \frac{2\pi n\, e^{\gamma A}}{A^2\gamma^2 + \pi^2 n^2}$$

so that we find

$$P(S \ge T) = \sum_n e^{-\lambda_n T - \gamma x}(H_n(\gamma, A) - H_n(\gamma, 0)) \sin \frac{n\pi x}{A}$$

From this we compute the expected stopping time. The probability distribution of S is

$$p(S = T) = \sum_n \lambda_n e^{-\lambda_n T - x\gamma}(H_n(\gamma, A) - H_n(\gamma, 0)) \sin \frac{n\pi x}{A}$$

so that

$$E(S) = \sum_n \frac{1}{\lambda_n} e^{-x\gamma}(H_n(\gamma, A) - H_n(\gamma, 0)) \sin \frac{n\pi x}{A}$$

Assume now that the experiment is truncated at a certain fixed time T. Then the modified stopping time is

$$S' = \begin{cases} S & \text{if } S \leq T \\ T & \text{if } S \geq T \end{cases}$$

Taking into account that

$$\int_0^T \lambda t e^{-\lambda t} dt = -\frac{(\lambda t + 1)e^{-\lambda t}}{\lambda}\bigg|_0^T = \frac{1}{\lambda}(1 - (T\lambda + 1)e^{-T\lambda})$$

we find

$$E(S') = P(S \geq T)T + \sum_n \frac{1}{\lambda_n}(1 - (T\lambda_n + 1)e^{-\lambda_n T})e^{-x\gamma}(H_n(\gamma, A) - H_n(\gamma, 0)) \sin \frac{n\pi x}{A}$$

3. The probability that the particle hits $t = T$, $x \leq y$ before hitting the upper or lower boundary

In this case we must use (2.1) with

$$\psi(x) = \begin{cases} 1 & x \leq y \\ 0 & x > y \end{cases}$$

Thus we find

$$P(x, t) = \sum_n e^{-\lambda_n T} e^{\lambda_n t - \gamma x}(H_n(\gamma, y) - H_n(\gamma, 0)) \sin \frac{n\pi x}{A}$$

So the probability that a particle starting at $(0, x)$ hits the interval $[(T, 0), (T, y)]$ before hitting the boundary is

$$\sum_n e^{-\lambda_n T - \gamma x}(H_n(\gamma, y) - H_n(\gamma, 0)) \sin \frac{n\pi x}{A}$$

4. The probability that the particle touches the upper boundary before time T

We must solve the boundary value problem for P with $P(0, t) = P(x, T) = 0$, $P(A, t) = 1$.

We first look for a $Q(x, t)$ solution indendent of t which satisfies $Q(0, t) = 0$, $Q(A, t) = 1$. Such a solution is of the form

$$C + De^{-2\gamma x}$$

for suitable constants C, D. The boundary conditions give the following constraints

$$C + D = 0$$
$$C + De^{-2\gamma A} = 1$$

which leads to

$$C = -\frac{1}{e^{-2\gamma A}-1}$$
$$D = \frac{1}{e^{-2\gamma A}-1}$$

Hence the solution has the form

$$Q(x,t) = \frac{e^{-2\gamma x}-1}{e^{-2\gamma A}-1}$$

Consider now

$$P'(x,t) = P(x,t) - Q(x,t)$$

Then $P'(0,t) = P'(A,t) = 0$ for $t \le A$ and

$$P'(x,T) = -\frac{e^{-2\gamma x}-1}{e^{-2\gamma A}-1}$$

so that using (2.1)

$$P'(x,t) = -\frac{1}{e^{-2\gamma A}-1}\sum_n e^{-\lambda_n T} e^{\lambda_n t-\gamma x}\sin\frac{n\pi x}{A}\left[\frac{2}{A}\int_0^A e^{\gamma y}\sin\frac{n\pi y}{A}(e^{-2\gamma y}-1)\,dy\right]$$

So the ultimate solution is

$$P(x,t) = \frac{e^{-2\gamma x}-1}{e^{-2\gamma A}-1} - \frac{1}{e^{-2\gamma A}-1}\sum_n e^{-\lambda_n T} e^{\lambda_n t-\gamma x}\sin\frac{n\pi x}{A}\left[\frac{2}{A}\int_0^A e^{\gamma y}\sin\frac{n\pi y}{A}(e^{-2\gamma y}-1)\,dy\right]$$

which can also be written as

$$P(x,t) = \frac{e^{-2\gamma x}-1}{e^{-2\gamma A}-1} - \frac{1}{e^{-2\gamma A}-1}\sum_n e^{-\lambda_n T} e^{\lambda_n t-\gamma x}\sin\frac{n\pi x}{A}(H_n(-\gamma,A)-H_n(\gamma,A))$$

Using the definition of $H_n(\gamma, A)$ this becomes
(4.1)

$$\begin{aligned}
P(x,t) &= \frac{e^{-2\gamma x}-1}{e^{-2\gamma A}-1} - \frac{1}{e^{-2\gamma A}-1}\sum_n (-(-1)^n) e^{-\lambda_n T} e^{\lambda_n t-\gamma x}\frac{2\pi n(e^{-\gamma A}-e^{\gamma A})}{A^2\gamma^2+\pi^2 n^2}\sin\frac{n\pi x}{A} \\
&= \frac{e^{-2\gamma x}-1}{e^{-2\gamma A}-1} - e^{\gamma A}\sum_n e^{-\lambda_n T} e^{\lambda_n t-\gamma x}\frac{2\pi n}{A^2\gamma^2+\pi^2 n^2}\sin\frac{n\pi(A-x)}{A} \\
&= \frac{e^{-2\gamma x}-1}{e^{-2\gamma A}-1} - \sum_n e^{-\lambda_n T} e^{\lambda_n t-\gamma x} H(\gamma,A)\sin\frac{n\pi x}{A}
\end{aligned}$$

So the probability that a particle starting at $(0, x)$ hits the interval $[(0, A), (T, A)]$ before hitting the lower boundary is.

$$\frac{e^{-2\gamma x}-1}{e^{-2\gamma A}-1} - \sum_n e^{-\lambda_n T-\gamma x} H_n(\gamma,A)\sin\frac{n\pi x}{A}$$

Put $T = \infty$ get that the probability that the particle hits $[(0, A), (\infty, A)]$ is

$$\frac{e^{-2\gamma x}-1}{e^{-2\gamma A}-1}$$

hence the probability that it hits $[(T, A), (\infty, A)]$ is

$$\sum_n e^{-\lambda_n T-\gamma x} H_n(\gamma,A)\sin\frac{n\pi x}{A}$$

5. The probability that the particle touches the lower boundary before time T

To get the probability that the particle touches the lower boundary first we have to make the substitutions $x \mapsto A - x$, $\mu \mapsto -\mu$ (and hence $\gamma \mapsto -\gamma$). Note that

$$H_n(-\gamma, A) = (-1)^n e^{-\gamma A} H_n(\gamma, 0)$$

So the probability that a particle starting at $(0, x)$ hits the interval $[(0,0),(T,0)]$ before hitting the upper boundary is.

$$\begin{aligned}\frac{e^{2\gamma(A-x)}-1}{e^{2\gamma A}-1} - &\sum_n e^{-\lambda_n T + \gamma(A-x)}(-1)^n e^{-\gamma A} H_n(\gamma, 0)\sin\frac{n\pi(A-x)}{A} \\ &= \frac{e^{2\gamma(A-x)}-1}{e^{2\gamma A}-1} + \sum_n e^{-\lambda_n T - \gamma x} H_n(\gamma, 0)\sin\frac{n\pi x}{A}\end{aligned}$$

6. The probability that the particle passes below (T, y).

Combining everything we get that the probability that a particle starting in $(0, x)$ leaves the rectangle $[0, T] \times [0, A]$ in a point below (T, y) where $x, y \in [0, A]$ is given by

$$\boxed{\frac{e^{2\gamma(A-x)}-1}{e^{2\gamma A}-1} + \sum_n e^{-\lambda_n T - \gamma x} H_n(\gamma, y)\sin\frac{n\pi x}{A}} \tag{6.1}$$

References

1. D. Siegmund, *Sequential analysis. Tests and confidence intervals.*, Springer, 1985.