1. General analysis

Assume we have i.i.d. random variables $(X_i)_{i\geq 1}$ with distribution $p_\theta(x)$. We are interested in the likelihood function after n observations.

$$L_\theta(x_1,\ldots,x_n)=\prod_{i=1}^n p_\theta(x_i)$$

Hence the log-likelihood is given by

$$\mathrm{LL}_\theta(x_1,\ldots,x_n)=\sum_{i=1}^n \log p_\theta(x_i)$$

We will approximate LL_θ by a quadratic function when θ is close to 0.

$$\mathrm{LL}_\theta(x_1,\ldots,x_n)\cong\sum_{i=1}^n p_0(x_i)+\theta\left(\sum_{i=1}^n\frac{\partial\log p_\theta(x_i)}{\partial\theta}\right)_{\theta=0}+\frac{\theta^2}{2}\left(\sum_{i=1}^n\frac{\partial^2\log p_\theta(x_i)}{\partial\theta^2}\right)_{\theta=0}$$

The first term does not depend on θ and hence is unimportant. The last term is small so we will replace it by its expectation value in 0 which is the $-n$ times the Fisher information in 0. So we get

$$\mathrm{LL}_\theta(x_1,\ldots,x_n)\sim\theta\sum_{i=1}^n S_0(x_i)-\frac{n\theta^2}{2}I_0$$

where S_θ is the scoring statistic

$$S_\theta=\frac{\partial\log p_\theta(X)}{\partial\theta}$$

Note

$$I_\theta=E_\theta(S_\theta^2)$$

We will put $Z=\sum_i S_{0,i}$.

1.1. Intermediate 2-SPRT.

Suppose we now perform a 2-SPRT with parameters $\theta_0<\theta_2<\theta_1$, A, B (θ_2 is where we want to minimize the expected duration of the test). The test stops whenever

$$\mathrm{LL}_{\theta_0}-\mathrm{LL}_{\theta_2}<\log A\ \text{(accept H1)}\qquad\text{or}$$
$$\mathrm{LL}_{\theta_1}-\mathrm{LL}_{\theta_2}<\log B\ \text{(accept H0)}$$

In other words whenever

$$(\theta_0-\theta_2)Z-\frac{n}{2}I_0(\theta_0^2-\theta_2^2)<\log A\qquad\text{or}$$
$$(\theta_1-\theta_2)Z-\frac{n}{2}I_0(\theta_1^2-\theta_2^2)<\log B$$

Or equivalently

$$Z>\frac{\log A^{-1}}{\theta_2-\theta_0}+\frac{n}{2}I_0(\theta_0+\theta_2)\qquad\text{or}$$
$$Z<-\frac{\log B^{-1}}{\theta_1-\theta_2}+\frac{n}{2}I_0(\theta_1+\theta_2)$$

Thus the test continues if

$$-\frac{\log B^{-1}}{\theta_1-\theta_2}+\frac{n}{2}I_0(\theta_1+\theta_2)<Z<\frac{\log A^{-1}}{\theta_2-\theta_0}+\frac{n}{2}I_0(\theta_0+\theta_2)$$

The maximal duration of the test is given by the solution to

$$-\frac{\log B^{-1}}{\theta_1-\theta_2}+\frac{n}{2}I_0(\theta_1+\theta_2)=\frac{\log A^{-1}}{\theta_2-\theta_0}+\frac{n}{2}I_0(\theta_0+\theta_2)$$

which may be solved by

$$n=\frac{2}{I_0(\theta_1-\theta_0)}\left(\frac{\log A^{-1}}{\theta_2-\theta_0}+\frac{\log B^{-1}}{\theta_1-\theta_2}\right)$$

1.2. **General 2-SPRT.** Let us now assume that $\theta_2<\theta_0$. Now the test stops whenever

$$Z<-\frac{\log A^{-1}}{\theta_0-\theta_2}+\frac{n}{2}I_0(\theta_0+\theta_2) \qquad \text{or}$$
$$Z<-\frac{\log B^{-1}}{\theta_1-\theta_2}+\frac{n}{2}I_0(\theta_1+\theta_2)$$

and hence the continuation region is

$$Z>-\frac{\log A^{-1}}{\theta_0-\theta_2}+\frac{n}{2}I_0(\theta_0+\theta_2) \qquad \text{and}$$
$$Z>-\frac{\log B^{-1}}{\theta_1-\theta_2}+\frac{n}{2}I_0(\theta_1+\theta_2)$$

This is now a 1-sided boundary.

2. Sanity check in case $X\sim N(\theta,1)$

We now assume $X\sim N(\theta,1)$, $A=B$, $\theta_0=-\delta$, $\theta_1=\delta$, $\theta_2=0$. We have

$$S_0(x)=\left(\frac{\partial-\frac{1}{2}(x-\theta)^2}{\partial\theta}\right)_{\theta=0}=x$$
$$I_0=E_0(S_0(X)^2)=1$$

Thus the continuation region of the test is

$$-\frac{\log A^{-1}}{\delta}+\frac{n}{2}\delta<Z<\frac{\log A^{-1}}{\delta}-\frac{n}{2}\delta$$

or equivalently

$$|Z|<\delta^{-1}\log A^{-1}-\frac{1}{2}n\delta$$

which is precisely as stated in Lorden.

3. Application to the BayesElo model

Let $L(x)$ be the logistic function

$$L(x)=\frac{1}{1+e^{-\beta x}}$$

where $\beta=(\log 10)/400$. $L(x)$ satisfies the differential equation

$$L'(x)=\beta L(x)(1-L(x))$$

as well as the functional equation

$$L(x)+L(-x)=1$$

In the BayesElo model the outcome probabilities are respectively

$$\begin{aligned} P_e(\text{win}) &= L(e-d) \\ P_e(\text{draw}) &= L(e+d) - L(e-d) \\ P_e(\text{loss}) &= 1 - L(e+d) \end{aligned}$$

where e is the elo-difference and d is the "draw elo" parameter (which we assume constant). One has the following nice identity

$$P_e(\text{draw}) = \lambda P_e(\text{win}) P_e(\text{loss}) \tag{1}$$

with

$$\lambda = e^{2\beta d} - 1$$

We now compute the scoring function for $e = 0$.

$$\begin{aligned} S_0(\text{win}) &= \left. \frac{L'(e-d)}{L(e-d)} \right|_{e=0} \\ &= \beta(1 - L(-d)) \\ &= \beta L(d) \end{aligned}$$

$$\begin{aligned} S_0(\text{loss}) &= \left. \frac{-L'(e+d)}{1 - L(e+d)} \right|_{e=0} \\ &= -\beta L(d) \end{aligned}$$

From (1) we then obtain

$$S_0(\text{draw}) = 0$$

(alternatively we may use the standard fact that $E_0(S_0) = 0$).

We conclude

$$\begin{aligned} I_0 &= E_0(S_0^2) \\ &= \beta^2 (P_0(\text{win}) + P_0(\text{loss})) L(d)^2 \\ &= 2\beta^2 L(-d) L(d)^2 \end{aligned}$$

Put $Z_0 = Z/(\beta L(d))$. The continuation region is now

$$-\frac{\log B^{-1}}{e_1 - e_2} + \frac{n}{2} 2\beta^2 L(-d) L(d)^2 (e_1 + e_2) < \beta Z_0 L(d) < \frac{\log A^{-1}}{e_2 - e_0} + \frac{n}{2} 2\beta^2 L(-d) L(d)^2 (e_0 + e_2)$$

More concretely

$$-\frac{\log B^{-1}}{(e_1 - e_2)\beta L(d)} + n\beta L(-d) L(d)(e_1 + e_2) < Z_0 < \frac{\log A^{-1}}{(e_2 - e_0)\beta L(d)} + n\beta L(-d) L(d)(e_0 + e_2)$$

Thus in Anderson's notation

$$c_2 + d_2 n < Z_0 < c_1 + d_1 n$$

with

$$\begin{aligned} c_1 &= \frac{\log A^{-1}}{(e_2 - e_0)\beta L(d)} \\ d_1 &= \beta L(-d) L(d)(e_0 + e_2) \\ c_2 &= -\frac{\log B^{-1}}{(e_1 - e_2)\beta L(d)} \\ d_2 &= \beta L(-d) L(d)(e_1 + e_2) \end{aligned}$$