ON GLOBAL DEFORMATION QUANTIZATION IN THE ALGEBRAIC CASE

MICHEL VAN DEN BERGH

Abstract. We give a proof of Yekutieli's global algebraic deformation quantization result which does not rely on the choice of local sections of the bundle of affine coordinate systems. Instead we use an argument inspired by algebraic De Rham cohomology.

Contents

1. Introduction and motivation

This paper has grown out of an attempt to understand [25, 37]. These papers deal with deformation quantization in an algebraic setting. After some consideration we decided no harm would be done by writing down our own account. For simplicity we restrict ourselves to infinitesimal deformations. The extension to formal deformations is routine.

Kontsevich's fundamental idea is that quantization of Poisson brackets should take place in the setting of *twisted presheaves*. To explain this let X be a separated quasi-compact scheme over a field k. Choose an affine covering $\mathcal{U} = \{U_1, \dots, U_n\}$ of X. For $J \subset \{1, \dots, n\}$ define $U_J = \cap_{j\in J} U_j$. A twisted presheaf of k-algebras on $\mathcal{U}$ is a collection of k-algebras $\mathcal{A}(U_J)$ together with restriction maps $\rho_{J,J'} : \mathcal{A}(U_J) \to \mathcal{A}(U_{J'})$ for $J \subset J'$ which are compatible with compositions up to an explicit inner automorphism. In addition the units defining these inner automorphisms should

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The author is a director of research at the FWO.

satisfy a natural cocycle conditions for triple compositions. The motivation is that under suitable flatness conditions one may define a category of quasi-coherent sheaves over a twisted presheaf.

Assume now that X is a smooth and separated over $k = \mathbb{C}$. For (l, m) an Artinian local k-algebra with residue field k, Kontsevich constructs in [25] a "quantization" arrow

$$\begin{gathered}\{\text{Poisson brackets on } X \text{ with coefficients in } m\}/\cong \\ \to \\ \{\text{Flat } l\text{-deformations of } \mathcal{O}_X \text{ in the category of twisted presheaves}\}/\cong\end{gathered} \tag{1.1}$$

To make the connection with the deformation theory of abelian categories [28, 29] note that in [27] Tor Lowen constructs a natural bijection

$$\begin{gathered}\{\text{Flat } l\text{-deformations of } \mathcal{O}_X \text{ in the category of twisted presheaves}\}/\cong \\ \leftrightarrow \\ \{\text{Flat } l\text{-deformations of } \mathrm{Qch}(\mathcal{O}_X)\}/\cong\end{gathered}$$

Our aim is to explain (1.1). This explanation will make it clear what the obstruction is against reversing the arrow.

Let us recall some basic constructions. The complex of sheaves of poly-differential operators $\mathcal{D}_X^{\mathrm{poly},\cdot}$ is defined as follows. For an open U of X, $\mathcal{D}_X^{\mathrm{poly},n}(U)$ is given by the multilinear maps $\mathcal{O}_U^{\otimes n} \to \mathcal{O}_U$ which are differential operators in each argument. $\mathcal{D}_X^{\mathrm{poly},\cdot}$ is equipped with the standard Hochschild differential and Gerstenhaber bracket. In this way $\mathcal{D}_X^{\mathrm{poly}}[1]$ becomes a sheaf of DG-Lie algebras.

Likewise $\mathcal{T}_X^{\mathrm{poly},\cdot}$ is defined as the graded sheaf on X whose sections of degree n on an open U are given by the multilinear maps $\mathcal{O}_U^{\otimes n} \to \mathcal{O}_U$ which are fully anti-symmetric and derivations in each argument. When equipped with the Schouten-Nijenhuis bracket and trivial differential $\mathcal{T}_X^{\mathrm{poly}}[1]$ also becomes a DG-Lie algebra.

The key result in algebraic deformation quantization is the following

Theorem 1.1. [37, Thm 0.2] *There is an isomorphism*

$$\mathcal{T}_X^{poly,\cdot}[1] \cong \mathcal{D}_X^{poly,\cdot}[1] \tag{1.2}$$

in the homotopy category of sheaves of DG-Lie algebras. Furthermore if X has a system of parameters $(x_i)_i$ then the resulting map on homology

$$\mathcal{T}_X^{poly,\cdot} \to H^\cdot(\mathcal{D}_X^{poly,\cdot})$$

is given by the HKR-formula.

$$\partial_{i_1} \wedge \cdots \wedge \partial_{i_n} \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \partial_{i_{\sigma(1)}} \otimes \cdots \otimes \partial_{i_{\sigma(n)}}$$

where $\partial_i = \partial_{x_i}$.

We will give a self contained proof of this result in this paper. We mimic to some extent Yekutieli's arguments and we use many of his technical contributions. However there is a substantial simplification that we do not have to choose (local) sections of the bundle of affine coordinate systems and thus we avoid the machinery of simplicial sections. Instead we use an argument inspired by algebraic De Rham cohomology (see §6.6 below). We believe this idea is new even in the classical case of Fedosov quantization. It follows in particular that (1.2) is compatible with automorphisms of X.

An analogue of Theorem 1.1 has been proved in the complex analytic case in [9]. The proof in loc. cit. does not extend immediately to the algebraic case as it depends on the choice of a global connection. In [13] the authors prove a version of Theorem 1.1 using operadic methods which is uniformly valid for the C^∞, algebraic and complex analytic cases. In [10] we will give yet another approach to these results.

Let us now explain how Proposition 1.1 is relevant to deformation theory. For a sheaf of DG-Lie algebras $\mathcal{G}$ on a topological space X one may take its derived global sections $R\Gamma(X,\mathcal{G})^{\text{tot}}$ which is also a DG-Lie algebra and which is canonically isomorphic to $R\Gamma(X,\mathcal{G})$ as complexes. In particular the formation of $R\Gamma(X,\mathcal{G})^{\text{tot}}$ is compatible with quasi-isomorphisms. One possible construction using pro-hypercoverings is outlined in Appendix B. A different construction follows from Hinich's model structure on the category of presheaves of algebras over an operad [21]. See §B.10. If $\mathcal{G}$ is quasi-coherent and X is separated then $R\Gamma(X,\mathcal{G})^{\text{tot}}$ is given by applying the Thom-Sullivan normalization (see Appendix A) to the cosimplicial DG-Lie algebra associated to an affine covering of X (see [22, 23]). We note that it will be clear below that only the properties of the functor $R\Gamma(X,-)^{\text{tot}}$ matter, not its actual construction.

Applying $R\Gamma(X,-)^{\text{tot}}$ to (1.2) we obtain in particular an isomorphism in the homotopy category of DG-Lie algebras

$$R\Gamma(X,\mathcal{T}_X^{\text{poly},\cdot}[1])^{\text{tot}} \cong R\Gamma(X,\mathcal{D}_X^{\text{poly},\cdot}[1])^{\text{tot}} \tag{1.3}$$

Let $\mathfrak{g}$ be a DG-Lie algebra. The *Maurer-Cartan equation* in $m\otimes_k \mathfrak{g}_1$ is given by

$$d\pi + \frac{1}{2}[\pi,\pi] = 0 \tag{1.4}$$

There is a a natural action on the solutions of this equation by the "gauge" group $\exp(m\otimes\mathfrak{g}_0)$. It is well known that the set of equivalence classes of solutions to the Maurer-Cartan equation is invariant under quasi-isomorphisms.

If A is a k-algebra then it is well known that the flat l-deformations of A correspond to solutions of the Maurer-Cartan equation in $m\otimes_k \mathbf{C}(A)[1]$ where $\mathbf{C}(A)$ is the Hochschild complex of A (equipped with the Gerstenhaber bracket). Similar results for abelian and linear categories were proved in [28, 29].

Let $\mathcal{U}$ be as above and let $\mathfrak{u}$ be the linear category with objects $\emptyset \subsetneq J \subset \{1,\dots,n\}$ and

$$\mathfrak{u}(J,J') = \begin{cases} \operatorname{Hom}_{\mathcal{O}_X}(j_*\mathcal{O}_{U_J}, j_*\mathcal{O}_{U_{J'}}) = \mathcal{O}_X(U_{J'}) & \text{if } J\subset J' \\ 0 & \text{otherwise} \end{cases}$$

where $j: U_J \to X$ denotes the inclusion map. We prove (Theorem 3.1 below)

$$R\Gamma(X,\mathcal{D}_X^{\text{poly},\cdot}[1])^{\text{tot}} \cong \mathbf{C}(\mathfrak{u})[1] \tag{1.5}$$

Since deformations of $\mathfrak{u}$ are readily seen to correspond to deformations of $\mathcal{O}_X$ as twisted presheaf, and vice versa (see [27]), it follows from (1.3) and (1.5) that we have bijections.

$$\{\text{Solutions to the MC equation in } R\Gamma(X, \mathcal{T}_X^{\text{poly},\cdot}[1])^{\text{tot}} \otimes_k m\}/\cong$$
$$\leftrightarrow$$
$$\{\text{Solutions to the MC equation in } R\Gamma(X, \mathcal{D}_X^{\text{poly},\cdot}[1])^{\text{tot}} \otimes_k m\}/\cong \tag{1.6}$$
$$\leftrightarrow$$
$$\{\text{Flat } l\text{-deformations of } \mathcal{O}_X \text{ in the category of twisted presheaves}\}/\cong$$

By Proposition B.8.1 below there is a canonical map

$$\Gamma(X, \mathcal{T}_X^{\text{poly},\cdot})[1] \to R\Gamma(X, \mathcal{T}_X^{\text{poly},\cdot}[1])^{\text{tot}} \tag{1.7}$$

and this is an isomorphism provided

$$H^i(X, \wedge^j \mathcal{T}_X) = 0 \qquad \text{for } i > 0 \tag{1.8}$$

The solutions to the Maurer-Cartan equation in $\Gamma(X, \mathcal{T}_X^{\text{poly},\cdot})$ are the global Poisson brackets on X. Thus combining (1.6) with (1.7) we now obtain the arrow (1.1) and we see that it is a bijection if (1.8) holds.

Using similar ideas (see Proposition 4.1) one proves that if X is proper there is a bijection

$$\{\text{Solutions to the MC equation in } \oplus_{i,j}\Gamma(X^{\text{an}}, \mathcal{T}^{i,0}_{X^{\text{an}}} \otimes_{\mathcal{O}^{\infty}_{X^{\text{an}}}} \Omega^{0,j}_{X^{\text{an}}})[1] \otimes_k m\}/\cong$$
$$\leftrightarrow$$
$$\{\text{Flat } l\text{-deformations of } \mathcal{O}_X \text{ in the category of twisted presheaves}\}/\cong$$

thereby making the connection with the work of Barannikov and Kontsevich [4]. The last bijection allows one, through the work of Gualtieri [16, §5.3], to associate a category of coherent sheaves to an infinitesimal deformation of X^{an} as *generalized complex manifold*. Generalized complex manifolds form a common generalization of complex and symplectic manifolds and as such are important for mirror symmetry. It is not known in general how to define a (derived?) category of coherent sheaves over a generalized complex manifold. In the case of a symplectic manifold this should be some variant of the Fukaya category.

Other papers relevant for algebraic deformation quantization are [7, 6, 30]. [6] is especially interesting as it discusses Fedosov quantization in positive characteristic. This falls totally outside the reach of methods based on DG-Lie algebras and the Maurer-Cartan equation.

We now give a quick outline of the current paper. In §3 we explain the connection between poly-differential operators and the Hochschild complex of schemes. In §4 we discuss the application to the analytic case mentioned above.

The proof of Theorem 1.1 uses crucially infinite dimensional formal schemes. We discuss the relevant topological notions in §5.

In §6 we use formal schemes to give an account of formal geometry in the algebraic case. See also [37, §5]. Theorem 6.6.1 is our crucial acyclicity result for the bundle of affine coordinate systems.

In §7 we present a reminder on DG-Lie and L_∞-algebras. An important notion is the twist of an L_∞-morphism by a solution of the Maurer-Cartan equation (which I learnt from Yekutieli). We also discuss descent for L_∞-morphisms under an algebraic group action and its compatibility with twisting. This is used to descend constructions on the bundle of local coordinate systems to the bundle of affine local coordinate systems.

In §8 we have a new look at poly-differential operators and poly-vector fields. We introduce Kontsevich's local L_∞-quasi-isomorphism and remind the reader of its properties. An interesting remark is that the linearity property (P3) thought to be essential for globalization actually follows from (P5).

Finally in §9 we prove Theorem 1.1.

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2. Notations and conventions

For simplicity of exposition we assume throughout that our base field is algebraically closed of characteristic zero (and usually $\mathbb{C}$). It is clear that with sufficient care one can get by with weaker hypotheses.

Many of the objects we use are equipped with some kind of topology, but if an object is introduced without a specified topology we assume that it is equipped with the discrete topology.

If an object carries a natural grading then all constructions associated to it are implicitly performed in the graded context. This implies in particular to completions.

3. Going from poly-differential operators to the Hochschild complex

The main result of this section (Theorem 3.1) was used in the introduction.

Let k be a field. If $\mathfrak{u}$ is a k-linear category then the Hochschild complex of $\mathfrak{u}$ is defined as

$$\mathbf{C}^n(\mathfrak{u}) = \prod_{U_0,\ldots,U_p\in \mathrm{Ob}(\mathfrak{u})} \mathrm{Hom}_k(\mathfrak{u}(U_{p-1},U_p)\otimes_k \cdots \otimes_k \mathfrak{u}(U_0,U_1),\mathfrak{u}(U_0,U_p)) \tag{3.1}$$

with the usual differential.

It is well-known that the Hochschild complex of a linear category has a lot of "higher structure". In particular it is a DG-Lie algebra when equipped with the Gerstenhaber bracket. This is the structure we will use below.

The Hochschild complex of a linear category is contravariantly functorial for fully faithful functors $\mathfrak{v}\to\mathfrak{u}$. The resulting map $\mathbf{C}(\mathfrak{u})\to\mathbf{C}(\mathfrak{v})$ will be called the restriction map.

Assume now that k has characteristic zero and let X be a smooth separated scheme over k. It will be convenient to use the notations $D^{\mathrm{poly},\cdot}(U)=\mathcal{D}_X^{\mathrm{poly},\cdot}(U)$ for $U\subset X$ open and $D^{\mathrm{poly},\cdot}(R)=D^{\mathrm{poly},\cdot}(X)$ for $X=\operatorname{Spec} R$, and similarly for poly-vector fields.

Assume first that $X=\operatorname{Spec} R$ is affine. We obtain an inclusion of complexes

$$D^{\mathrm{poly},\cdot}(R)[1]\to\mathbf{C}(R)[1] \tag{3.2}$$

which is compatible with the DG-Lie algebra structures on both sides. In [36] it is shown that (3.2) is a quasi-isomorphism.

We have isomorphisms

$$D^{\mathrm{poly},\cdot}(R)[1] \cong D^{\mathrm{poly},\cdot}(X)[1] \cong R\Gamma(X, \mathcal{D}_X^{\mathrm{poly},\cdot})^{\mathrm{tot}}[1] \tag{3.3}$$

The first isomorphism is a tautology and the second one follows from Proposition B.8.1 below.

Now we drop the restriction that X is affine. Select an affine open covering $\mathcal{U} = \{U_1, \dots, U_n\}$ of X and let the associated notations be as in the introduction.

Theorem 3.1. (1) *There is an isomorphism in the homotopy category of DG-Lie algebras*

$$R\Gamma(X, \mathcal{D}_X^{\mathrm{poly}})^{\mathrm{tot}}[1] \cong \mathbf{C}(\mathfrak{u})[1] \tag{3.4}$$

(2) *If $X = \operatorname{Spec} R$ is affine and $\mathcal{U} = \{X\}$ then* (3.4) *coincides with the composition of* (3.2) *and the inverse of* (3.3).

Proof. If $p : \mathfrak{h} \to \mathfrak{g}$ is a map of cosimplicial DG-Lie algebras then as in §B.7 below we say that p is a weak equivalence if p induces a quasi-isomorphism between the cochain complexes $C^*(\mathfrak{h})$ and $C^*(\mathfrak{g})$.

We first prove (1). Let $\mathrm{Ch}^o(\mathcal{U}, \mathcal{D}_X^{\mathrm{poly}})[1]$ be the ordered Čech cosimplicial DG-Lie algebra associated to $\mathcal{D}_X^{\mathrm{poly}}[1]$ and the cover $\mathcal{U}$ (see §B.9). We will construct a cosimplicial DG-Lie algebra $\mathcal{C}[1]$ together with weak equivalences

$$\mathrm{Ch}^o(\mathcal{U}, \mathcal{D}_X^{\mathrm{poly}})[1] \to \mathcal{C}[1] \leftarrow \mathbf{C}(\mathfrak{u})[1] \tag{3.5}$$

where we view $\mathbf{C}(\mathfrak{u})[1]$ as a constant cosimplicial DG-Lie algebra.

Applying the Thom-Sullivan normalization functor $N(-)^{\mathrm{TS}}$ (see Appendix A) we obtain isomorphisms in the homotopy category of DG-Lie algebras

$$\begin{aligned} R\Gamma(X, \mathcal{D}_X^{\mathrm{poly}})^{\mathrm{tot}}[1] \xrightarrow{(1)} N(\mathrm{Ch}^o(\mathcal{U}, \mathcal{D}_X^{\mathrm{poly}})[1])^{\mathrm{TS}} \to \\ N(\mathcal{C}[1])^{\mathrm{TS}} \leftarrow N(\mathbf{C}(\mathfrak{u})[1])^{\mathrm{TS}} \xleftarrow{(2)} \mathbf{C}(\mathfrak{u})[1] \end{aligned} \tag{3.6}$$

Here arrow (1) is obtained from §B.9 below and (2) is obtained from (A.1). The composed isomorphism in (3.6) yields part (1) of the theorem. Part (2) will follow from the construction of $\mathcal{C}$.

So now we concentrate on (3.5). For clarity we will sometimes omit the shift [1] in the formulas. For $\emptyset \neq J \subset I$ let $\mathfrak{u}_J$ be the full subcategory of $\mathfrak{u}$ spanned by J', $J' \supset J$. Since $\mathfrak{u}_{J'} \to \mathfrak{u}_J$ is fully faithful it follows that there are restriction maps

$$\mathbf{C}(\mathfrak{u}_J) \to \mathbf{C}(\mathfrak{u}_{J'}) \tag{3.7}$$

for $J' \supset J$. Furthermore it follows from [29, Lemma 7.5.2] that the restriction map

$$\mathbf{C}(\mathfrak{u}_J) \to \mathbf{C}(\mathcal{O}_{U_J}) \tag{3.8}$$

is a quasi-isomorphism.

Define

$$\mathcal{C}^m = \prod_{j_0 \leq \cdots \leq j_m} \mathbf{C}(\mathfrak{u}_{\{j_0, \dots, j_m\}})$$

We make $\mathcal{C} = (\mathcal{C}^m)_m$ into a cosimplicial DG-Lie algebra using the restriction maps (3.7).

Consider $\mathbf{C}(\mathfrak{u})$ as a constant cosimplicial DG-Lie algebra. We claim that the restriction map

$$\mathbf{C}(\mathfrak{u}) \to \mathcal{C}$$

is a weak equivalence. To this end it is sufficient to check that we obtain a quasi-isomorphism on the corresponding (totalized) normalized cochain complexes.

The normalized cochain complex of $\mathcal{C}$ it the total complex of a double complex with columns

$$\mathcal{C}^{(m)} = \prod_{j_0<\cdots<j_m} \mathbf{C}(\mathfrak{u}_{\{j_0,\dots,j_m\}})$$

Write $\mathfrak{u}_\emptyset = \mathfrak{u}$ and $\mathcal{C}^{(-1)} = \mathbf{C}(\mathfrak{u})$. We have to show that the total complex associated to the double complex

$$0 \to \mathcal{C}^{(-1)} \to \mathcal{C}^{(0)} \to \cdots \to \mathcal{C}^{(n)} \to 0 \tag{3.9}$$

is acyclic. We do this by showing that it is a long exact sequence of complexes.

$\mathbf{C}(\mathfrak{u}_J)$ is a direct product of abelian groups of the form

$$\mathrm{Hom}_k(\mathcal{O}_X(U_{J_1}) \times \cdots \times \mathcal{O}_X(U_{J_t}), \mathcal{O}_X(U_{J_t})) \tag{3.10}$$

for $J \subset J_0 \subset \cdots \subset J_t$. It follows that the summands in (3.9) corresponding to a given sequence $J_0, \cdots, J_t$ are parametrized by $J \subset J_0$. Since the signs are the usual alternating ones it follows easily that for the horizontal differential (3.9) is a sum of acyclic complexes. This proves what we want.

Our next aim is to construct a cosimplicial map

$$c : \mathrm{Ch}^o(\mathcal{U}, \mathcal{D}_X^{\mathrm{poly}}) \to \mathcal{C}$$

We do this by combining maps

$$c_J : D^{\mathrm{poly}}(U_J) \to \mathbf{C}(\mathfrak{u}_J) \tag{3.11}$$

If $d \in D^{\mathrm{poly,t}}(U_J)$ then d is a differential operator in

$$\mathrm{Hom}(\mathcal{O}_X(U_J)^t, \mathcal{O}_X(U_J))$$

It follows that d extends uniquely to a differential operator with t arguments in (3.10). We define $c_J(d)$ as this extension.

Now we claim that c is a weak equivalence. To do this it is sufficient to show that the maps c_J are quasi-isomorphisms. Then it is sufficient that the composition

$$D^{\mathrm{poly}}(U_J) \to \mathbf{C}(\mathfrak{u}_J) \to \mathbf{C}(\mathcal{O}_{U_J})$$

of c_J with the quasi-isomorphism (3.8) is a quasi-isomorphism.

This composition in nothing but (3.2) for $R = \mathcal{O}(U_J)$ and hence we are done. □

Remark 3.2. It seems quite likely that the fact that $R\Gamma(X, \mathcal{D}_X)^{\mathrm{tot}}$ controls the deformation theory of $\mathcal{O}_X$ in the category of twisted presheaves follows also from Vladimir Hinich's descent theorem [19] given the fact that this is true if X is affine. One has to check that the global deformation functor is given by gluing the local deformation fuctors. Note that for this to work one should view these deformation functors as taking values in 2-groupoids.

4. The analytic case

The main result in this section (Proposition 4.1) was used in the introduction. Assume now that $k = \mathbb{C}$ and X is a separated smooth proper scheme over k. Let

$\mathcal{O}^\infty_{X^{\mathrm{an}}}$ be the sheaf of C^∞-functions on X^{an} and let $\mathcal{T}^{i,0}_{X^{\mathrm{an}}}$ and $\Omega^{0,j}_{X^{\mathrm{an}}}$ be the C^∞-vector bundles respectively generated by the holomorphic vector fields and by the anti-holomorphic differential forms. Below we consider the sheaf of DG-Lie algebras

$$\bigoplus_{i,j} \mathcal{T}^{i,0}_{X^{\mathrm{an}}} \otimes_{\mathcal{O}^\infty_{X^{\mathrm{an}}}} \Omega^{0,j}_{X^{\mathrm{an}}}[1] \tag{4.1}$$

where the differential is obtained by linearly extending the differential on $\Omega^{0,\cdot}_{X^{\mathrm{an}}}$ and the Lie bracket is obtained by linearly extending the Lie bracket on $\mathcal{T}^{\cdot,0}_{X^{\mathrm{an}}}$.

Proposition 4.1. *There is an isomorphism*

$$R\Gamma(X, \mathcal{T}^{\mathrm{poly},\cdot}_X)^{\mathrm{tot}}[1] \cong \bigoplus_{i,j} \Gamma(X^{\mathrm{an}}, \mathcal{T}^{i,0}_{X^{\mathrm{an}}} \otimes_{\mathcal{O}^\infty_{X^{\mathrm{an}}}} \Omega^{0,j}_{X^{\mathrm{an}}})[1]$$

in the homotopy category of DG-Lie algebras.

Proof. We first prove that there is an isomorphism

$$R\Gamma(X, \mathcal{T}^{\mathrm{poly},\cdot}_X)^{\mathrm{tot}}[1] \to R\Gamma(X^{\mathrm{an}}, \mathcal{T}^{\mathrm{poly},\cdot}_{X^{\mathrm{an}}})^{\mathrm{tot}}[1] \tag{4.2}$$

Choose an affine covering $\mathcal{U} = \{U_1, \ldots, U_n\}$ for X. By Lemma B.9.1 and the fact that affine varieties are Stein we have

$$R\Gamma(X, \mathcal{T}^{\mathrm{poly},\cdot}_X)^{\mathrm{tot}} = N(\mathrm{Ch}^o(\mathcal{U}, \mathcal{T}^{\mathrm{poly},\cdot}_X))^{TS}$$
$$R\Gamma(X^{\mathrm{an}}, \mathcal{T}^{\mathrm{poly},\cdot}_{X^{\mathrm{an}}})^{\mathrm{tot}} = N(\mathrm{Ch}^o(\mathcal{U}, \mathcal{T}^{\mathrm{poly},\cdot}_{X^{\mathrm{an}}}))^{TS}$$

The obvious map

$$\mathrm{Ch}^o(\mathcal{U}, \mathcal{T}^{\mathrm{poly},\cdot}_X) \to \mathrm{Ch}^o(\mathcal{U}, \mathcal{T}^{\mathrm{poly},\cdot}_{X^{\mathrm{an}}})$$

yields (4.2). To prove that (4.2) is an isomorphism it is sufficient to prove that it induces an isomorphism on cohomology. The cohomology on the left and on the right are respectively given by

$$H^i(X, \wedge^j \mathcal{T}_X) \qquad \text{and} \qquad H^i(X^{\mathrm{an}}, \wedge^j \mathcal{T}_{X^{\mathrm{an}}})$$

These are are equal because of GAGA.

We now note that the sheaf of DG-Lie algebras (4.1) is an acyclic resolution for $\mathcal{T}^{\mathrm{poly},\cdot}_{X^{\mathrm{an}}}$ (the $\bar\partial$-resolution). Using Lemma B.9.1 we have

$$R\Gamma(X, \mathcal{T}^{\mathrm{poly},\cdot}_{X^{\mathrm{an}}})^{\mathrm{tot}}[1] \cong \bigoplus_{i,j} \Gamma(X^{\mathrm{an}}, \mathcal{T}^{i,0}_{X^{\mathrm{an}}} \otimes_{\mathcal{O}^\infty_{X^{\mathrm{an}}}} \Omega^{0,j}_{X^{\mathrm{an}}})[1] \tag{4.3}$$

This concludes the proof. □

Remark 4.2. In [4] Barannikov and Kontsevich show that if X is Calabi-Yau then $\bigoplus_{i,j} \Gamma(X^{\mathrm{an}}, \mathcal{T}^{i,0}_{X^{\mathrm{an}}} \otimes_{\mathcal{O}^\infty_{X^{\mathrm{an}}}} \Omega^{0,j}_{X^{\mathrm{an}}})[1]$ is isomorphic in the homotopy category of DG-Lie algebra to the vector space

$$\bigoplus_{i,j} H^i(X, \wedge^j \mathcal{T}_X)[1]$$

with zero differential and Lie bracket. It follows that for (l, m) an Artinian local ring with residue field k the l-deformations of $\mathrm{Qch}(\mathcal{O}_X)$ correspond to elements of

$$m \otimes_k \left(\bigoplus_{i+j=2} H^i(X, \wedge^j \mathcal{T}_X) \right)$$

5. Topological notions

Below we will naturally encounter topological rings and modules. Rather than using Yekutieli's category of Dir Inv-abelian groups [37] we work in the classical setting of filtered topological abelian groups. Below we list the few facts we need.

5.1. **Topological abelian groups.**

Below we will encounter exclusively linear topological abelian groups. I.e. topological abelian groups equipped with a topology such that 0 has a system of open neighborhoods consisting of subgroups.

The following trivial lemma is often used implicitly.

Lemma 5.1.1. *Assume that V is a linear topological abelian group and W is an open subgroup. Then W is also closed and the quotient topology on V/W is discrete.*

Proof. W is the complement of the open set $\bigcup_{w\not\in W}(w+W)$ and hence W is closed. □

From now on we assume in addition that there is a countable basis of neighborhoods of 0. It is convenient to take for such a basis a descending chain of subgroups $V = F_0V \supset F_1V \supset \cdots$, constructed by taking successive intersections in an arbitrary linear basis indexed by the natural numbers.

We define the completion of a linear topological abelian group V as

$$\hat{V} = \operatorname*{proj\,lim}_p V/F_pV$$

It is easy to see that this is the same as the usual completion using Cauchy sequences. $\hat{V}$ is a linear topological abelian group with neighborhood basis of 0 given by the $(F_pV)\hat{}$. V is complete if the map $V \to \hat{V}$ is an homeomorphism.

If V, W is are linear topological abelian groups then we make $V \otimes_{\mathbb{Z}} W$ into a linear topological abelian group by selecting as system of neighborhoods for 0 the images of the abelian groups $F_pV \otimes_{\mathbb{Z}} W + V \otimes_{\mathbb{Z}} F_pW$. The completed tensor product $V \hat{\otimes}_{\mathbb{Z}} W$ is the completion of $V \otimes_{\mathbb{Z}} W$ for this topology.

Since now the categories of linear topological abelian groups and complete linear topological abelian groups are tensor categories we can define vector spaces, rings, modules etc... in them.

5.2. **Filtered linear topological abelian groups.**

It will be necessary to use filtered linear topological abelian groups. A filtered linear topological abelian group is by definition an abelian group V, equipped with a filtration $F^\cdot$ such that each F^mV is equipped with a linear topology with the property that the inclusion maps $F^mV \to F^{m+1}V$ are continuous. The filtration is $F^\cdot$ is considered part of the structure of a filtered linear topological abelian group.

We say that V is complete if each F^mV is complete. The completion of V is defined as $\bigcup (F^mV)\hat{}$. The category of filtered and complete filtered linear topological abelian groups have the obvious structure of a monoidal category, so we can define rings, modules etc... in them.

Example 5.2.1. The ring of differential operators of $k[[t]]$, which is equal to $k[[t]][\partial_t]$ has an obvious structure of complete filtered linear topological ring.

5.3. **Finite adic rings.** In this section all rings are commutative. We will need to complete non-noetherian rings. This is a somewhat dangerous operation for the following reason. Let T be a ring with an ideal I and put $\hat{T} = \operatorname{proj\,lim} T/I^n$. If we equip all T/I^n with the discrete topology then $\hat{T}$ becomes a topological ring. However in general we will not have $T/I \cong \hat{T}/I\hat{T}$. The reason for this is that $I\hat{T}$ is not closed, i.e. $I\hat{T} \neq (I\hat{T})\hat{} = \hat{I}$. So we should replace $I\hat{T}$ by $\hat{I}$. Unfortunately this does not resolve all our problems since in general $\hat{I^n} \neq (\hat{I})^n$. So we still don't have an isomorphism $R/I^n \cong R/(\hat{I})^n$. The following example clarifies this.

Example 5.3.1. Let $T = k[x_1, x_2, \ldots]$ be the polynomial ring in infinitely many variables over k and let $I = (x_1, x_2, \ldots)$. Then $\hat{T} = T[[x_1, x_2, \ldots]]$.

The ideals $(I^n)\hat{}$ are topologically generated by the monomials $x_{i_1} \cdots x_{i_n}$ with $i_1 \leq i_2 \leq \cdots \leq i_n$. The following element shows

$$x_1 + x_2x_3 + x_4x_5x_6 + \cdots$$

that $\hat{I}$ is not generated by $x_1, x_2, \ldots$ as an ordinary ideal.

We now show that $\hat{I}^2 \neq (I^2)\hat{}$. If $f \in \hat{I}^2$ then the partial derivatives $f_i = \partial f/\partial x_i$ are in a finitely generated ideal in $\hat{I}$ (exercise). Consider the element

$$f = x_1^2 + x_2^3 + x_3^4 + \cdots$$

By working modulo x_m for $m > n$ we see by looking at heights that an ideal containing the f_i needs at least n generators. Since n is arbitrary this means that the f_i cannot be contained in a finitely generated ideal.

Luckily all problems go away if we consider completions at finitely generated ideals.

Definition 5.3.2. An *adic* ring [11, 0.7.1.9] is a linear topological ring such that

$$T = \operatorname{proj\,lim}_n T/I^n \tag{5.1}$$

where of course T/I^n is equipped with the discrete topology.

An alternative way of stating (5.1) is by saying that I^n is a fundamental system of *open* neighborhoods of 0 and that the topology on T is separated and complete. An ideal I with this property is called an *ideal of definition* of T. Note that if T has the discrete topology then T is adic and the zero ideal is an ideal of definition.

The following definition is non-standard but convenient.

Definition 5.3.3. A *finite adic ring* is an adic ring with a finitely generated ideal of definition.

If T is an adic ring with ideal of definition I and M is a topological T module then we say that M is adic if

$$M = \operatorname{proj\,lim}_n M/I^nM$$

where as usual M/I^nM is equipped with the discrete topology. It is clear that this definition does not depend on the choice of I.

Theorem 5.3.4. *Assume that T is a ring with an ideal I such that I/I^2 is finitely generated and let M be a T-module. Let $\hat{T}$ and $\hat{M}$ be respectively the completions of T and M for the I-adic topology. Then we have the following:*

(1) *$\hat{T}$ is an adic ring with ideal of definition $I\hat{T}$.*
(2) *If $f_1, \dots, f_d$ are lifts in I of generators of I/I^2 then the images of $f_1, \dots, f_d$ in $\hat{T}$ are generators of $I\hat{T}$. In particular $\hat{T}$ is finite adic.*
(3) *$\hat{M}$ is adic.*
(4) *We have $(I^n M)\hat{} = I^n \hat{M}$*
(5) *The canonical map $M/I^n M \to \hat{M}/I^n \hat{M}$ is an isomorphism.*

Proof. This is a slight refinement of [11, Prop. 0.7.2.7]. □

The category of adic topological rings has tensor products. More concretely let $C \to A$, $C \to B$ be continuous maps of linear topological rings such that A, B have finitely generated ideals of definition I, J. Then it is easy to see that the topology on $A \otimes_C B$ induced from $A \otimes B$ has a finitely generated defining ideal which is given as the image of $K = I \otimes B + A \otimes J$. We define $A \hat{\otimes}_C B$ as the completion of $A \otimes_C B$ for this topology. Note that for this definition the topology on C does not have to be adic. Also note that if A, B are finite adic then so is $A \hat{\otimes}_C B$.

Convention 5.3.5. *Let A, B be k-algebras. It will often happen below that there is given some finitely finitely generated ideal $I \subset A \otimes B$ and that we are interested in the I-adic completion of $A \otimes B$ at I. To avoid confusion we will write this completion as $A \hat{\boxtimes} B$. A we use a similar convention for modules. If M, N are respectively A, B-modules then $M \hat{\boxtimes} N$ denotes the I-adic completion of $M \otimes N$ at I.*

If T is an adic ring then we will write $\mathrm{Adic}(T)$ for the additive category of adic T-modules. Note that any T-linear map between objects in $\mathrm{Adic}(T)$ is automatically continuous. If T has a finitely generated defining ideal then completion defines a left adjoint to the forgetful functor

$$\mathrm{Adic}(T) \to \mathrm{Mod}(T)$$

5.4. **Differentials.** If M is a T-module then the symmetric group acts on $M^{\otimes n}$ (the n-fold tensor product of M over T) by permuting the factors.

We put

$$\bigwedge\nolimits_T^i M = \operatorname{coker}\left(\bigoplus_{\sigma \in S_n} M^{\otimes n} \xrightarrow{\sum_\sigma \phi_\sigma} M^{\otimes n} \right)$$

where ϕ_σ acts by $1 - (-1)^{\mathrm{sign}(\sigma)}\sigma$ on the summand indexed by σ. It follows immediately that $\bigwedge_T^i M$ is compatible with base change in T.

If T is an R-algebra then as usual we write

$$\Omega^i_{T/R} = \bigwedge\nolimits_T^i \Omega_{T/R}$$

and we call the collection of T-modules $\Omega^i_{T/R}$, together with its natural differential the *relative De Rham complex* $\Omega^{\cdot}_{T/R}$ of T/R.

If T is an adic R-algebra then we write.

$$\Omega^{i,\mathrm{cont}}_{T/R} = (\Omega^i_{T/R})\hat{}$$

As derivations are automatically continuous with respect to the I-adic topology (for I an ideal of definition of T) the differential on $\Omega^{\cdot}_{T/R}$ lifts to $\Omega^{\cdot,\mathrm{cont}}_{T/R}$. We call the resulting complex the *continuous relative De Rham complex* of T/R.

Assume now that T is finite adic. By Theorem 5.3.4 the $\Omega^{i,\mathrm{cont}}_{T/R}$ are adic T-modules.

The following formula is convenient

Proposition 5.4.1. *We have*

$$\Omega^{i,\mathrm{cont}}_{T/R} = \operatorname*{proj\,lim}_n \Omega^i_{(T/I^n)/R}$$

Proof. We have a standard exact sequence

$$I^n/I^{2n} \to \Omega_{T/R}/I^n\Omega_{T/R} \to \Omega_{(T/I^n)/R} \to 0$$

Hence modulo essentially zero systems we have an isomorphism between the inverse systems $(\Omega_{T/R}/I^n\Omega_{T/R})_n$ and $(\Omega_{(T/I^n)/R})_n$. It is easy to see that we obtain from this an isomorphism between the inverse systems $(\Omega^i_{T/R}/I^n\Omega^i_{T/R})_n$ and $(\Omega^i_{(T/I^n)/R})_n$, modulo essentially zero systems. Taking the inverse limit proves what we want. □

Example 5.4.2. (A. Yekutieli) Consider $T = k[[t]]$. As $\Omega_{T/k}$ is compatible with localization we have $(\Omega_{T/k})_t = \Omega_{T_t/k}$. Since $T_t = k((t))$ is a field of infinite transcendence degree over k, it follows that $\Omega_{T/k}$ is a very large object. On the other hand $\Omega^{\mathrm{cont}}_{T/k}$ is equal to $T dt$.

For $M \in \mathrm{Mod}(T)$ let us denote by $\mathrm{Der}^i_R(T, M)$ the set of anti-symmetric multilinear maps $T \otimes_R \cdots \otimes_R T \to M$ with i arguments, which are derivations in each of their arguments.

Clearly $\Omega^i_{T/R}$ represents the functor

$$\mathrm{Der}^i_R(T, -) : \mathrm{Mod}(T) \to \mathrm{Mod}(R) :$$

Similarly $\Omega^{i,\mathrm{cont}}_{T/R}$ also represents $\mathrm{Der}^i_R(T, -)$ but now considered as a functor $\mathrm{Adic}(T) \to \mathrm{Mod}(R)$.

Proposition 5.4.3. *Assume that T is a ring with a finitely generated ideal I. Let $\widehat{(-)}$ stand for I-adic completion. Then the canonical map*

$$\Omega^i_{T/R}/I^n\Omega^i_{T/R} \to \Omega^{i,cont}_{\hat{T}/R}/I^n\Omega^{i,cont}_{\hat{T}/R}$$

is an isomorphism. In particular

$$(\Omega^i_{T/R})\hat{} = \Omega^{i,cont}_{\hat{T}/R}$$

Proof. $\Omega^i_{T/R}/I^n\Omega^i_{T/R}$ represents the functor

$$\mathrm{Der}^i_R(T, -) : \mathrm{Mod}(T/I^n) \to \mathrm{Mod}(R)$$

and likewise $\Omega^{i,\mathrm{cont}}_{\hat{T}/R}/I^n\Omega^{i,\mathrm{cont}}_{\hat{T}/R}$ represents the functor

$$\mathrm{Der}_R(\hat{T}, -) : \mathrm{Mod}(\hat{T}/I^n\hat{T}) \to \mathrm{Mod}(R)$$

It is easy to see that these functors are naturally isomorphic if we make the identification $\mathrm{Mod}(T/I^n) \cong \mathrm{Mod}(\hat{T}/I^n\hat{T})$ (using Theorem 5.3.4). □

5.5. **Formal schemes.** A standard reference for the basic material on formal schemes is [11]. Since the formal schemes we use are not noetherian, we recall the basics. Let I be an ideal in a ring T. For $f \in T/I$ let $\tilde{f} \in T$ stand for an arbitrary lift of f. Then it is easy to see that the completion of $T_{\tilde{f}}$ with respect to the ideal $I_{\tilde{f}}$ is (canonically) independent of the choice of $\tilde{f}$. We will write $\hat{T}_f$ for this completed localization. If M is a T-module then $\hat{M}_f$ can be define likewise.

If T is finite adic then follows from Theorem 5.3.4 that all $\hat{T}_f$ are finite adic as well and furthermore all $\hat{M}_f$ are adic $\hat{T}_f$-modules.

An alternative definition for $\hat{M}_f$ is the following. f defines an open subset $D(f)$ of $\operatorname{Spec} T/I^n = \operatorname{Spec} T/I$. Then $\hat{M}_f$ is given by the global sections of

$$\operatorname{proj}\lim_n (M/I^n M)\tilde{}\mid D(f)$$

(where as usual $(-)\tilde{}$ denotes the quasi-coherent sheaf associated to a module).

Now we sheaffify these constructions. Recall the technically useful fact that an inverse limit of sheaves can be computed as presheaves.

If T is a finite adic ring with finitely generated ideal of definition we define

$$\operatorname{Spc} T = (\operatorname{Spec} T/I, \operatorname{proj}\lim_n (T/I^n)\tilde{})$$

for an ideal of definition I. So $\operatorname{Spc} T$ is a topologically ringed $(\mathcal{T}, \mathcal{O}_{\mathcal{T}})$ with $\mathcal{T} = \operatorname{Spec} T/I$. It is clear that the definition of $\operatorname{Spc} T$ is independent of I and

$$(D(f), \mathcal{O}_{\mathcal{T}}|D(f)) = \operatorname{Spc} \hat{T}_f$$

We use a special type of formal scheme. For the full definition see [11].

Definition 5.5.1. An *finite adic affine formal scheme* is a topologically ringed space which is isomorphic to $\operatorname{Spc} T$ for a finite adic ring T.

Let T now be a finite adic ring with finitely generated ideal of definition I. Put $(\mathcal{T}, \mathcal{O}_{\mathcal{T}}) = \operatorname{Spc} T$. If M is an adic T-module then we define

$$M^{\triangle} = \operatorname{proj}\lim_n (M/I^n M)\tilde{}$$

Thus $M^{\triangle}$ is a sheaf of topological $\mathcal{O}_{\mathcal{T}}$-modules such that

$$M^{\triangle} \mid D(f) = (\hat{M}_f)^{\triangle}$$

$\mathcal{O}_{\mathcal{T}}$ itself contains a sheaf of ideals $\mathcal{I} = I^{\triangle}$ such that

$$\operatorname{proj}\lim_n \mathcal{O}_{\mathcal{T}}/\mathcal{I}^n = \mathcal{O}_{\mathcal{T}}$$

We call $\mathcal{I}$ an ideal of definition of $\mathcal{O}_{\mathcal{T}}$. Being an ideal of definition is a local property (see [11, Prop 10.3.5]).

Chaining the various definitions and using Proposition 5.4.3 we also find

$$(\Omega^{i,\mathrm{cont}}_{T/R})^{\triangle} \mid D(f) = (\Omega^{i,\mathrm{cont}}_{\hat{T}_f/R})^{\triangle}$$

Definition 5.5.2. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a topologically ringed space. We say that $\mathcal{X}$ is a finite adic formal scheme if $\mathcal{X}$ is locally a finite adic affine topologically ringed space.

As usual we call $\mathcal{O}_{\mathcal{X}}$ the structure sheaf of $\mathcal{X}$. We say that a topological $\mathcal{O}_{\mathcal{X}}$-module is adic if it is locally of the form M^{Δ}. If $X \to Y$ is a map of a finite adic formal scheme to a scheme then we define $\Omega^{i,\text{cont}}_{Y/X}$ as the $\mathcal{O}_{\mathcal{X}}$-module is which is locally, on open affine formal subschemes $\operatorname{Spc}(T)$, of the form $\Omega^{i,\text{cont}}_{T/R}$.

Morphisms between formal schemes are by definition morphisms of locally topologically ringed spaces [11, Def 10.4.5]. If S, T are finite adic rings then

$$\operatorname{Hom}_{\text{formal schemes}}(\operatorname{Spc} S, \operatorname{Spc} T) = \operatorname{Hom}_{\text{topological rings}}(T, S)$$

[11, Prop 10.2.2].

Clearly the category of (affine) schemes is a full subcategory of the category of finite adic (affine) formal schemes.

The category of finite adic formal schemes has fibered products [11, Prop 10.7.3]. As usual it is sufficient to construct these for affine formal schemes. If $C \to A$, $C \to B$ are continuous maps of adic rings then we put

$$\operatorname{Spc} A \times_{\operatorname{Spc} C} \operatorname{Spc} B = \operatorname{Spc}(A \hat{\otimes}_C B)$$

Suppose X is a scheme and Y is a closed subscheme defined by a quasi-coherent ideal $\mathcal{I}$ which is (locally) of finite type. Then $\hat{X}_Y$ (or simply $\hat{X}$) is the finite adic formal scheme whose underlying space is Y and whose structure sheaf is

$$\mathcal{O}_{\hat{X}_Y} = \hat{\mathcal{O}}_{X,Y} = \operatorname{proj}\lim_n \mathcal{O}_X/\mathcal{I}^n$$

6. Formal geometry

Many ideas in this section are taken from [37, §5]. However we put more emphasis on the language of formal schemes.

6.1. **Basic definitions.** Everything will be over an algebraically closed base field k of characteristic zero. Fix an integer d. For a finite adic scheme Y with locally finitely generated ideal of definition $\mathcal{I}$ we let $Y[[t_1, \dots, t_n]]$ be the finite adic formal scheme which is the completion of $Y \times \mathbb{A}^d_k$ at the ideal $\mathcal{I} + (t_1, \dots, t_d)$. The inclusion/projection $Y = Y \times \{0\} \hookrightarrow Y \times A^d_k \to Y$ yields a canonical projection map

$$p_Y : Y[[t_1, \dots, t_n]] \to Y \tag{6.1}$$

with section

$$i_Y : Y \to Y[[t_1, \dots, t_n]] \tag{6.2}$$

If $Y = \operatorname{Spc} S$ is affine then

$$Y[[t_1, \dots, t_d]] \cong \operatorname{Spc} S[[t_1, \dots, t_d]]$$

We like to view (6.1) as an infinite dimensional vector bundle over Y with zero-section given by (6.2).

Let FSch/k be the category of finite adic formal schemes over k.

Proposition 6.1.1. *The functor*

$$\operatorname{FSch}/k \to \operatorname{FSch}/k : Y \mapsto Y[[t_1, \dots, t_d]]$$

has a right adjoint.

Proof. Let $X \in \mathrm{FSch}/k$. We need to show that the contravariant functor

$$\Phi : \mathrm{FSch}/k \to \mathrm{Set} : Y \mapsto \mathrm{Hom}_{\mathrm{FSch}/k}(Y[[t_1,\ldots,t_d]], X) \tag{6.3}$$

is representable.

Since maps of finite adic formal schemes are compatible with gluing we reduce to $Y = \mathrm{Spc}\, S$, $X = \mathrm{Spc}\, R$. So we may work with the category of finite adic affine formal schemes, or equivalently, the category of finite adic rings. Thus

$$\Phi(S) = \mathrm{Hom}(R, S[[t_1,\ldots,t_d]])$$

Let $R^{\mathbf{d},\circ}$ be the k-algebra generated by symbols $f_{\underline{i}}$ for $f \in R$ and $\underline{i} = (i_1,\ldots,i_d)$ with relations

$$\begin{aligned} (f+g)\tilde{} &= \tilde{f} + \tilde{g} \\ (fg)\tilde{} &= \tilde{f}\tilde{g} \\ \tilde{\lambda} &= \lambda \qquad (\text{for } \lambda \in k) \end{aligned}$$

where $\tilde{f}$ is the generating function $\sum_{\underline{i}} f_{\underline{i}} t^{\underline{i}}$ with $t^{\underline{i}} = t_1^{i_1} \cdots t_d^{i_d}$. In particular there is a ring homomorphism

$$R \to R^{\mathbf{d},\circ} : f \mapsto f_{(0,\ldots,0)}$$

Let I be a finitely generated defining ideal for R and consider the ideal $J \subset R^{\mathbf{d},\circ}$ the ideal generated by $f_{(0,\cdots,0)}$ for $f \in I$. Then clearly J is also finitely generated. Let $R^{\mathbf{d}}$ be the completion of $R^{\mathbf{d},\circ}$ at J.

It is easy to see that we have a functorial isomorphism for any finite adic k-algebra S:

$$\mu : \mathrm{Hom}(R, S[[t_1,\ldots,t_d]]) \to \mathrm{Hom}(R^{\mathbf{d}}, S) \tag{6.4}$$

where μ is defined by

$$\sum_{\underline{i}} \mu(\phi)(f_{\underline{i}})\, t^{\underline{i}} = \phi(f)$$

Hence $R^{\mathbf{d}}$ represents Φ. □

Below we denote the right adjoint to $Y \mapsto Y[[t_1,\ldots,t_d]]$ by $X \mapsto X^{\mathbf{d}}$. The proof of the previous proposition shows that if X is affine then so is $X^{\mathbf{d}}$.

Now assume that $\phi : Y \to X$ is a map between k-schemes such that X is separated and of finite type. Then the graph $\Gamma_\phi = (\phi, \mathrm{id}_Y) : Y \to X \times Y$ of ϕ is closed and its defining ideal is of finite type. We let $\mathrm{Jet}_{\phi,\infty}$ be the completion of $Y \times X$ along Γ_ϕ. Thus $\mathrm{Jet}_{\phi,\infty}$ comes equipped with a map of formal schemes

$$\mathrm{Jet}_{\phi,\infty} \to X \times Y$$

An interesting special case is when ϕ is the inclusion of a closed point x in X. In that case

$$\mathrm{Jet}_{\phi,\infty} = \mathrm{Spc}\, \hat{\mathcal{O}}_{X,x} \tag{6.5}$$

It is easy to see that $\mathrm{Jet}_{\phi,\infty}$ is compatible with base extension in the sense that if there is a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\phi} \\ X & = & X \end{array}$$

then

$$\mathrm{Jet}_{\theta,\infty} = Z \times_Y \mathrm{Jet}_{\phi,\infty} \tag{6.6}$$

We will write

$$\mathrm{Jet}_{X,\infty} = \mathrm{Jet}_{\mathrm{id}_X,\infty}$$

We obtain in particular

$$Y \times_X \mathrm{Jet}_{X,\infty} = \mathrm{Jet}_{\phi,\infty}$$

Now let Y be a an arbitrary k-scheme. Fix a map

$$\phi : Y[[t_1, \dots, t_d]] \to X$$

We then get a commutative diagram

$$\begin{array}{ccc} Y & = & Y \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle (\phi\circ i_Y, \mathrm{id}_Y)} \\ Y[[t_1, \dots, t_d]] & \xrightarrow[(\phi, p_Y)]{} & X \times Y \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ Y & = & Y \end{array} \tag{6.7}$$

where the composition of the vertical arrows is the identity. Put $\phi_0 = \phi \circ i_Y$.

If we complete the middle arrow at Y we get a map.

$$\hat{\phi}_0 : Y[[t_1, \dots, t_d]] \to \mathrm{Jet}_{\phi_0,\infty} \tag{6.8}$$

Definition 6.1.2. Let the notations be as above. We say that ϕ is a local coordinate system parametrized by Y if $\hat{\phi}_0$ is an isomorphism of formal schemes.

Assume that ϕ is a local system of parameters and fix a k-point $y : \operatorname{Spec} k \to Y$. Pulling back (6.8) in the category of formal schemes and using (6.5)(6.6) we get an isomorphism of $k[[t_1, \dots, t_n]]$ and $\hat{\mathcal{O}}_{X,x}$ where $x = (\phi \circ i_Y)(y)$. In particular X is smooth at the image of x.

It follows that it is meaningless to talk about local coordinate systems for non-smooth schemes. So we now assume that X is smooth of dimension d over k.

Assume in addition that X has a system of parameters $x_1, \dots, x_d$. Let $\mathcal{I} \subset \mathcal{O}_{X\times X}$ be the defining ideal of the diagonal Δ. Then the sections $x_i' = x_i \otimes 1 - 1 \otimes x_i$ of $\mathcal{I}$ form a generating regular sequence of the ideal of definition $\hat{\mathcal{I}}$ of $\mathrm{Jet}_{X,\infty}$ (we are in the context of noetherian schemes and noetherian formal schemes so there are no subtleties). Invoking Proposition 6.1.3 below (for affine open subsets of $\mathrm{Jet}_{X,\infty}$) we find:

$$\mathrm{Jet}_{X,\infty} \cong \Delta[[x_1', \dots, x_d']]$$

and hence by base extension

$$\mathrm{Jet}_{\phi,\infty} \cong \Gamma_\phi[[x_1', \dots, x_d']] \tag{6.9}$$

Proposition 6.1.3. *Assume that we have a map $R \to T$ where R is a ring and T is finite adic. Assume that T has a defining ideal I generated by a regular sequence $x_1, \dots, x_n$ such that $T/I \cong R$ and such that the composition $R \to T \to R$ is an isomorphism. Then $T \cong R[[x_1, \dots, x_n]]$.*

Proof. Putting $\phi(x_i) = x_i$ defines a continuous map

$$\phi : R[[x_1, \dots, x_n]] \to T$$

Since in both rings the $(x_i)_i$ form a regular sequence this map becomes an isomorphism after taking associated graded rings. And since the topologies involved are separated and complete, this easily implies that ϕ is an isomorphism. □

Theorem 6.1.4. *The subfunctor*

$$\Phi^0 : \mathrm{Sch}/k \to \mathrm{Set} : Y \mapsto \{\textit{local coordinate systems on } X\}$$

of Φ *(as in* (6.3)*) is representable by an open subscheme* X^{coord} *of* $X^{\mathbf{d}}$ *which is still affine over* X*.*

Proof. As usual this is a local statement on X. Hence we may assume that X has a system of parameters $x_1, \dots, x_d$.

Assume given a local coordinate system on X, indexed by Y:

$$\phi : Y[[t_1, \dots, t_d]] \to X$$

Thus by the above discussion we obtain a map

$$\hat{\phi}_0 : Y[[t_1, \dots, t_d]] \to \Gamma_\phi[[x'_1, \dots, x'_d]] \tag{6.10}$$

We may write the pullbacks of the x'_i as $\sum_j a_{ij} t_j + \cdots$ for functions a_{ij} on Y. Put $\det \phi_0 = \det a_{ij}$. Then $\hat{\phi}_0$ is an isomorphism if and only if $\det \phi_0$ is a unit. Note that $\det \phi_0$ is well defined up to a unit.

Since it is easy to see that the formation of (6.10) is compatible with pullbacks this implies that Φ^0 is an open subfunctor of Φ. So it is representable by an open subscheme X^{coord} of $X^{\mathbf{d}}$. It is in fact represented by the open subset defined by $\det \theta_0$ for $\theta : X^{\mathbf{d}}[[t_1, \dots, t_d]] \to X$ the universal map. From the fact that $X^{\mathbf{d}}$ is affine over X, we deduce that X^{coord} is affine as well. □

Let $\theta : X^{\mathrm{coord}}[[t_1, \dots, t_d]] \to X$ be the universal local coordinate system on X. From Definition 6.1.2 we obtain a canonical isomorphism

$$\hat{\theta}_0 : X^{\mathrm{coord}}[[t_1, \dots, t_d]] \to \mathrm{Jet}_{\theta_0, \infty} \tag{6.11}$$

of finite adic X^{coord}-schemes.

If $X = \operatorname{Spec} R$ for a d-dimensional smooth k-algebra we write R^{coord} for the coordinate ring of X^{coord}. We obtain an isomorphism

$$R^{\mathrm{coord}} \mathbin{\hat{\boxtimes}} R \to R^{\mathrm{coord}}[[t_1, \dots, t_d]] : r \otimes f \mapsto r\tilde{f} \tag{6.12}$$

where following Convention 5.3.5 we let $R^{\mathrm{coord}} \mathbin{\hat{\boxtimes}} R$ be the completion of $R^{\mathrm{coord}} \otimes R$ at the kernel of the multiplication map

$$R^{\mathrm{coord}} \otimes R \to R^{\mathrm{coord}} : r \otimes f \mapsto r\tilde{f} \tag{6.13}$$

Example 6.1.5. It is instructive to understand the isomorphism (6.12) in the simplest possible case, namely when $R = k[x]$. In that case

$$R^{\mathrm{coord}} = k[x_0, x_1, \dots]_{x_1}$$

and (6.12) is given by

$$(k[x, x_0, x_1, \dots]_{x_1})\hat{} \to (k[t, x_0, x_1, \dots]_{x_1})\hat{} : x_i \mapsto x_i, x \mapsto \sum_{i \geq 0} x_i t^i \tag{6.14}$$

where the first completion is at the ideal $(x - x_0)$ and the second completion is at the ideal (t). To see directly that (6.14) is an isomorphism we look at associated graded rings. Putting $\delta = (x - x_0)$ the associated graded map to (6.14) is given by

$$(k[x_0, x_1, \dots]_{x_1})[\delta] \to (k[x_0, x_1, \dots]_{x_1})[t] : x_i \mapsto x_i, \delta \mapsto x_1 t$$

This is clearly an isomorphism.

6.2. **Groups and actions.** Recall that by definition for R a smooth d-dimensional k-algebra we have

$$\operatorname{Hom}(R^{\mathbf{d}}, S) \cong \operatorname{Hom}(R, S[[t_1, \dots, t_d]])$$

where S is an arbitrary finite adic k-algebra. According to Definition 6.1.2 a map $\phi : R^{\mathbf{d}} \to S$ represents a local coordinate system on $\operatorname{Spec} R$, parametrized by $\operatorname{Spc} S$ (i.e. an element of $\operatorname{Hom}(R^{\mathrm{coord}}, S)$) if the corresponding map $\phi : R \to S[[t_1, \dots, t_d]]$ induces an isomorphism

$$S \mathbin{\hat{\boxtimes}} R \to S[[t_1, \dots, t_d]] \tag{6.15}$$

where $S \mathbin{\hat{\boxtimes}} R$ is the completion of $S \otimes R$ at the kernel of the ideal $S \otimes R \to S : s \otimes r \mapsto s\phi_0(r)$ where ϕ_0 is the kernel of the composition

$$R \xrightarrow{\phi} S[[t_1, \dots, t_d]] \xrightarrow{t_i \mapsto 0} S$$

Lemma 6.2.1. *The functor which sends a finite adic k-algebra S to the group*

$$\operatorname{Aut}_S(S[[t_1, \cdots, t_d]]) \tag{6.16}$$

is representable by an affine finite adic formal k-scheme.

Proof. We sketch the proof which is similar to the proof of Proposition 6.1.1. Let A be the k-algebra generated by variables $z_{i,j_1,\dots,j_d}$ for $i = 1, \dots, d$, $j_l \geq 0$, localized at the determinant of the matrix z_{i,e_j} where $e_j = (0, \dots, 1, \dots, 0)$ has its 1 in the j'th position.

Let $\hat{A}$ be the completion of A at the ideal generated by $(z_{i,0,\dots,0})_i$. There is a bijection

$$\mu : \operatorname{Hom}(\hat{A}, S) \to \operatorname{Aut}_S(S[[t_1, \dots, t_d]])$$

defined by

$$\mu(\phi)(t_i) = \sum_{j_1,\dots,j_d} \phi(z_{i,j_1,\dots,j_d}) t_1^{j_1} \cdots t_d^{j_d}$$

Thus $\hat{A}$ represents (6.16) (as finite adic ring). Hence $\operatorname{Spc} \hat{A}$ represents (6.16) in the category of finite adic formal schemes. □

Below we denote the representing object of (6.16) by G. It is a group object in the category of formal schemes.

The canonical action of $G(S)$ on $\operatorname{Hom}(R, S[[t_1, \dots, t_d]])$ now defines an action on $\operatorname{Hom}(R^{\mathbf{d}}, S)$. Since this action is functorial in S we obtain an action of G on $R^{\mathbf{d}}$. Since $G(S)$ also acts on local coordinate systems it is clear that we obtain in addition an action of G on R^{coord}. Finally since everything is compatible with base change these actions globalize to the case of not necessarily affine d-dimensional smooth k-schemes.

Proposition 6.2.2. *Let X be a separated d-dimensional smooth k-scheme. Then the action of G on X^{coord} is free in the sense that*

$$G \times X^{\mathrm{coord}} \to X^{\mathrm{coord}} \times X^{\mathrm{coord}} : (g,x) \to (x, gx) \tag{6.17}$$

is a monomorphism.

Proof. For S finite adic we have to show that (6.17) induces an injection

$$G(S) \times X^{\mathrm{coord}}(S) \to X^{\mathrm{coord}}(S) \times X^{\mathrm{coord}}(S)$$

As usual we may reduce to the case that $X = \operatorname{Spec} R$ is affine. Then the statement amounts to proving that $G(S)$ acts freely on morphisms

$$R \to S[[t_1, \dots, t_d]]$$

defining a local coordinate systems parametrized by $\operatorname{Spc} S$. This follows from the existence of the isomorphism (6.15). □

The Lie algebra of G is defined as the kernel $G(k[\epsilon]) \to G(k)$. It can be naturally identified with the Lie algebra $\mathfrak{g}$ of derivations of $k[[t_1, \cdots, t_d]]$.

The following remarkable result is the main result of "formal geometry" [15]. It says that in a suitable sense X^{coord} is a principal homogeneous space over G. We will not explicitly use it however.

Proposition 6.2.3. *As before let X be a separated smooth k-scheme of dimension d. For $x \in X^{\mathrm{coord}}$ let $T_x(X^{\mathrm{coord}})$ be the tangent space at x, i.e. the set of maps $\operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X^{\mathrm{coord}}$ such that the composition $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X^{\mathrm{coord}}$ is x. Then the map $\mathfrak{g} \to T_x(X^{\mathrm{coord}})$ induced by the G-action on X^{coord} is an isomorphism of vector spaces.*

Proof. Since (6.17) is a monomorphism we obtain an injection

$$\mathfrak{g} \times T_x(X^{\mathrm{coord}}) \to T_x(X^{\mathrm{coord}}) \times T_x(X^{\mathrm{coord}})$$

we have to prove that this is a bijection. That is, if x_1, x_2 are $k[\epsilon]/(\epsilon^2)$-points of X^{coord} mapping x then there is an $g \in G(k[\epsilon]/(\epsilon^2))$, mapping to the identity in $G(k)$ such that $gx_1 = x_2$.

We may assume that $X = \operatorname{Spec} R$ is affine. Let x° be the image of x in X. Then x is given by a map

$$x : R \to k[[t_1, \dots, t_d]]$$

inducing an isomorphism

$$\hat{R} \to k[[t_1, \dots, t_d]]$$

where $\hat{R}$ is the completion of R at the maximal ideal of R defining x°.

Then x_1, x_2 are maps maps making the following diagram commutative

$$\begin{array}{ccc} R & \xrightarrow{x_1} & k[\epsilon]/(\epsilon^2)[[t_1, \dots, t_d]] \\ {\scriptstyle x_2}\downarrow & \searrow^{x} & \downarrow \\ k[\epsilon]/(\epsilon^2)[[t_1, \dots, t_d]] & \longrightarrow & k[[t_1, \dots, t_d]] \end{array} \tag{6.18}$$

Both x_1, x_2 induce $k[\epsilon]/(\epsilon^2)$-linear isomorphisms

$$\hat{R} \otimes k[\epsilon]/(\epsilon^2) \to k[\epsilon]/(\epsilon^2)[[t_1, \dots, t_d]]$$

From this it follows we can complete the diagram (6.18) with a $k[\epsilon]/(\epsilon^2)$-linear diagonal arrow $k[\epsilon]/(\epsilon^2)[[t_1,\dots,t_d]] \to k[\epsilon]/(\epsilon^2)[[t_1,\dots,t_d]]$. This is the required element of $G(k[\epsilon]/(\epsilon^2))$. □

The action of G on $R^{\mathbf{d}}$ may be differentiated to an action of $\mathfrak{g}$ on $R^{\mathbf{d}}$. The following proposition describes the nature of this action.

Proposition 6.2.4. *Let R be a smooth affine k-algebra of dimension d. For $f \in R$ let $\tilde{f} \in R^{\mathbf{d}}[[t_1,\dots,t_d]]$ be as in the proof of Proposition 6.1.1. For $v \in \mathfrak{g}$ let L_v be the action v on $R^{\mathbf{d}}[[t_1,\dots,t_d]]$ obtained by linearly extending the action of v on $k[[t_1,\dots,t_d]]$ (recall that $\mathfrak{g} = \operatorname{Der}_k(k[[t_1,\dots,t_d]])$). Let $L_{\bar{v}}$ be the action of v on $R^{\mathbf{d}}[[t_1,\dots,t_d]]$ by linearly extending the action of v on $R^{\mathbf{d}}$. Then we have*

$$L_{\bar{v}}(\tilde{f}) = -L_v(\tilde{f}) \tag{6.19}$$

Proof. We have the universal map

$$R \to R^{\mathbf{d}}[[t_1,\dots,t_d]] : f \mapsto \tilde{f}$$

which which is easily seen to be G-invariant (for G-acting trivially on R). Formula (6.19) expresses the fact that the differentiated G-action, given by $L_{\bar{v}} + L_v$ acts trivially on $\tilde{f}$ for $f \in R$. □

Example 6.2.5. It is again interesting to consider the simple case $R = k[x]$. We have

$$R^{\mathbf{d}} = k[x_0, x_1, \dots]$$

and

$$\mathfrak{g} = k[[t]]\partial_t$$

To compute the action of $\mathfrak{g}$ we note that $\mathfrak{g}$ has a k-linear topological basis given by $\delta_i = t^i\partial_t$. We have

$$[\delta_i, \delta_j] = (j-i)\delta_{i+j-1}$$

To compute the action of δ_i we use the method of proof of Proposition 6.2.4. Thus we use

$$\begin{aligned} 0 &= \delta_i(\sum_j x_j t^j) \\ &= \sum_j \delta_i(x_j)t^j + \sum_j x_j j t^{i+j-1} \\ &= \sum_j \delta_i(x_j)t^j + \sum_{j\ge i-1} (j-i+1)x_{j-i+1}t^j \\ &= \sum_{j<i-1} \delta_i(x_j)t^j + \sum_{j\ge i-1} (\delta_i(x_j) + (j-i+1)x_{j-i+1})t^j \end{aligned}$$

Thus it follows

$$\delta_i(x_j) = \begin{cases} 0 & \text{if } j < i-1 \\ -(j-i+1)x_{j-i+1} & \text{if } j \ge i-1 \end{cases}$$

or simply

$$\delta_i(x_j) = -(j-i+1)x_{j-i+1} \tag{6.20}$$

using the convention $x_j = 0$ for $j < 0$.

We obtain for $\alpha_i \in k$

$$\left(\sum_{i\geq 0} \alpha_i \delta_i\right) \cdot x_j = -\sum_{i\geq 0} \alpha_i (j-i+1) x_{j-i+1}$$

This is a finite sum so the action of $\mathfrak{g}$ on $R^{\mathbf{d}}$ is indeed well defined.

It is clear from (6.20) that the action of δ_0 of $R^{\mathbf{d}}$ cannot be exponentiated (i.e. the action of e^{δ_0} on x_i does not yield a finite sum). So the $\mathfrak{g}$-action cannot be exponentiated to a group action on $R^{\mathbf{d}}$. *However* the results in this section show that $\mathfrak{g}$ can be exponentiated to a group G *in the category of finite adic schemes.*

6.3. **Affine coordinate systems.** By restricting ourselves to linear coordinate changes we may view GL_d as a subgroup of G. The action (6.17) now restricts to a free action

$$\mathrm{GL}_d \times X^{\mathrm{coord}} \to X^{\mathrm{coord}}$$

Since X^{coord}/X is affine we may define the scheme $X^{\mathrm{aff}} = X^{\mathrm{coord}}/\mathrm{GL}_d$. Following our usual practice we write R^{aff} for the coordinate ring of $(\operatorname{Spec} R)^{\mathrm{aff}}$.

The advantage of X^{aff} over X^{coord} is the following property.

Proposition 6.3.1. *X^{aff} is a bundle of (∞-dimensional) affine spaces over X.*

Proof. Assume that $X = \operatorname{Spec} R$ is affine and that R has a system of parameters $x_1, \dots, x_d$. Consider the closed subscheme Y of X^{coord} whose S-points are given by maps

$$\phi : R \to S[[t_1, \dots, t_d]]$$

such that $\phi(x_i) = a_i + t_i + \cdots$ for certain $a_i \in S$. It is clear that the obvious map $\mathrm{GL}_d \times Y \to X$ defines a bijection on S-points and hence is an isomorphism. Thus $Y \cong X/\mathrm{GL}_d$.

Using the fact that $R/k[x_1, \dots, x_d]$ is etale (and hence formally etale) we see that any diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \uparrow & & \uparrow{\scriptstyle t_i \to 0} \\ k[x_1, \dots, x_d] & \longrightarrow & S[[t_1, \dots, t_d]] \end{array}$$

may be completed uniquely with a diagonal arrow $R \to S[[t_1, \dots, t_d]]$.

It is now clear that sending ϕ to ϕ_0 together with the coefficients of $\phi(x_i)$ of the terms of degree ≥ 2 defines a bijection between Y and the S-points of the product of $\operatorname{Spec} R$ with an infinite dimensional affine space. This proves what we want. $\square$

6.4. **The abstract formalism of Maurer Cartan forms.** This is an abstract section whose results will be employed in the next section. We consider

$$\mathfrak{g} = \mathrm{Der}_{k[[t_1, \dots, x_d]]}(k[[t_1, \dots, t_d]])$$

together with its natural topological Lie algebra structure. Let T be a finite adic k-algebra. Then the DG-Lie algebra

$$\Omega^{\cdot}_{T/k} \hat{\otimes}_k \mathfrak{g}$$

may be written as

$$\sum_i \Omega^{\cdot}_{T/k}[[t_1, \cdots, t_d]] \left[\frac{\partial}{\partial t_i}\right]$$

and hence its action on

$$\Omega^{\cdot}_{T/k}[[t_1,\dots,t_d]]$$

is clearly faithful.

We want to classify the derivations of degree one on $\Omega^{\cdot}_{T/k}[[t_1,\dots,t_d]]$ such that the natural map of algebras

$$\Omega^{\cdot}_{T/k} \to \Omega^{\cdot}_{T/k}[[t_1,\dots,t_d]]$$

becomes a map of DG-algebras. Below d is such a differential.

Clearly d is determined by the values $\omega_i = dt_i \in \Omega^1_{T/k}[[t_1,\dots,t_d]]$ or equivalently by the restriction of d to $k[[t_1,\dots,t_d]]$. This yields us a derivation

$$\delta : k[[t_1,\dots,t_d]] \to \Omega^1_{T/k}[[t_1,\dots,t_d]]$$

such that $\delta(t_i) = \omega_i$.

Put

$$\omega = \sum_i \omega_i \frac{\partial}{\partial t_i} \in \Omega^1_{T/R} \hat{\otimes}_k \mathfrak{g}$$

Then with a slight abuse of notation d may be written as

$$d = d_0 + \omega$$

where d_0 is the extension of the differential on $\Omega^{\cdot}_{T/R}$. The fact that $d^2 = 0$ translates into the identity

$$d_0 \circ \omega + \omega \circ d_0 + \omega \circ \omega = 0 \tag{6.21}$$

as operations on $\Omega^{\cdot}_{T/R}[[t_1,\dots,t_d]]$. The left hand side of this identity is the image of

$$d_0\omega + \frac{1}{2}[\omega,\omega]$$

in $\Omega^{\cdot}_{T/k} \hat{\otimes} \mathfrak{g}$. Hence (using faithfulness) (6.21) is nothing but the Maurer-Cartan equation

$$d_0\omega + \frac{1}{2}[\omega,\omega] = 0$$

in the DG-Lie algebra $\Omega^{\cdot}_{T/k} \hat{\otimes} \mathfrak{g}$ (compare with (1.4)).

6.5. **The Maurer-Cartan form on coordinate spaces.** Tensoring (6.12) on the left by the graded R^{coord}-module $\Omega^{\cdot}_{R^{\text{coord}}}$ and completing we obtain an isomorphism of graded commutative algebras.

$$\Omega^{\cdot}_{R^{\text{coord}}} \hat{\boxtimes} R \cong \Omega^{\cdot}_{R^{\text{coord}}}[[t_1,\dots,t_d]] : \eta \otimes f \mapsto \eta f \tag{6.22}$$

The DG-algebra structure on $\Omega^{\cdot}_{R^{\text{coord}}} \hat{\boxtimes} R$ now induces a DG-algebra structure on $\Omega^{\cdot}_{R^{\text{coord}}}[[t_1,\dots,t_d]]$ and thus according to the abstract discussion in §6.4 there is an associated Maurer-Cartan form

$$\omega_{\text{MC}} \in \Omega^1_{R^{\text{coord}}} \hat{\otimes} \mathfrak{g}$$

such that for $\eta \in \Omega^i_{R^{\text{coord}}}$, $f \in R$

$$(d\eta)\tilde{f} = (d + \omega_{\text{MC}})(\eta\tilde{f})$$

which is equivalent to

$$(d + \omega_{\text{MC}})(\tilde{f}) = 0 \tag{6.23}$$

The following lemma will be used below.

Lemma 6.5.1. *For $v \in \mathfrak{g}$ let $i_{\bar{v}}$ be the contraction on $\Omega^{\cdot}_{R^{\text{coord}}}$ with the derivation on R^{coord} induced by v (cfr (6.19)). Extend $i_{\bar{v}}$ to a map of degree -1 from $\Omega^{\cdot}_{R^{\text{coord}}} \hat{\otimes} \mathfrak{g}$ to itself. Then we have*

$$i_{\bar{v}}\omega_{MC} = 1 \otimes v \tag{6.24}$$

where both sides are elements of $R^{\text{coord}} \hat{\otimes} \mathfrak{g}$.

Proof. It is easy to see that for any $\omega \in \Omega^1_{R^{\text{coord}}} \hat{\otimes} \mathfrak{g}$ we have $(i_{\bar{v}}\omega)(\tilde{f}) = i_{\bar{v}}(\omega(\tilde{f}))$. Applying $i_{\bar{v}}$ to (6.23) and using this fact we obtain

$$(L_{\bar{v}} + i_{\bar{v}}\omega_{MC})(\tilde{f}) = 0$$

Or using (6.19)

$$(i_{\bar{v}}\omega_{MC})(\tilde{f}) = L_v(\tilde{f})$$

The operators on both sides are R^{coord}-linear. Since $R^{\text{coord}}[[t_1, \ldots, t_d]]$ is topologically generated by the $\tilde{f}$, $f \in R$ and R^{coord} (by the isomorphism (6.12)) we obtain as operators on $R^{\text{coord}}[[t_1, \ldots, t_d]]$

$$i_{\bar{v}}\omega_{MC} = L_v \tag{6.25}$$

Then (6.24) is the same equation as (6.25) but interpreted in $\Omega^{\cdot}_{R^{\text{coord}}} \hat{\otimes} \mathfrak{g}$ using faithfulness (see §6.4). □

6.6. **An acyclicity result.** Assume that X is a separated smooth k-scheme of dimension d. Let $\theta_0 : X^{\text{aff}} \to X$ be the canonical map. If X^{aff} were finite dimensional then the following result would follow trivially from the theory of algebraic De Rham cohomology [18] together with Proposition 6.3.1.

Theorem 6.6.1. *Put $J = \text{Jet}_{\theta_0,\infty}$. Then the canonical map*

$$\mathcal{O}_X \to \pi_*\Omega^{\cdot,cont}_{J/X}$$

is a quasi-isomorphism where $\pi : J \to X$ is the composition of the map $J \to X \times X^{\text{aff}}$ with the projection on the first factor.

Proof. Since this result is local on X we may assume $X = \operatorname{Spec} R$ and R has a system of parameters $x_1, \ldots, x_d$. Put $x'_i = \theta_0(x_i) \otimes 1 - 1 \otimes x_i$ and

Let $I \subset R^{\text{aff}} \otimes R$ be the kernel of the multiplication map $R^{\text{aff}} \otimes R \to R^{\text{aff}}$. Then $J = \operatorname{Spc} R^{\text{aff}} \hat{\boxtimes} R$ where $R^{\text{aff}} \hat{\boxtimes} R$ is the completion of $R^{\text{aff}} \otimes R$ at the ideal I and

$$\Gamma(X, \pi_*\Omega^{\cdot,\text{cont}}_{J/X}) = \Omega^{\cdot,\text{cont}}_{R^{\text{aff}}\hat{\boxtimes}R/R}$$

We have an R-linear isomorphism $R^{\text{aff}} \cong (R^{\text{aff}} \otimes R)/I$ (where on the right hand side R acts on the nose and on the left hand side it acts via the map θ_0). By Proposition 6.3.1 R^{aff}/R is formally smooth. Using formal smoothness we obtain a (non-canonical) R-linear splitting of the map $(R^{\text{aff}} \otimes R)/I^n \to (R^{\text{aff}} \otimes R)/I \cong R^{\text{aff}}$.

Taking the inverse limit over n we obtain a commutative diagram

$$\begin{array}{ccc} R^{\text{aff}} \hat{\boxtimes} R & \xrightarrow{r\otimes f \mapsto r\theta_0(f)} & R^{\text{aff}} \\ {\scriptstyle 1\otimes \text{id}_R}\uparrow & & \uparrow{\scriptstyle \theta_0} \\ R & = & R \end{array} \tag{6.26}$$

where the top map in (6.26) is (non-canonically) split as R-algebras.

Using Proposition 6.1.3 we obtain an isomorphism

$$R^{\text{aff}} \hat{\boxtimes} R \cong R^{\text{aff}}[[x'_1, \ldots, x'_d]] \tag{6.27}$$

of R-algebras.

According to Theorem 6.7.1 below the canonical map

$$\Omega^{\cdot}_{R^{\mathrm{aff}}/R} \to \Omega^{\mathrm{cont},\cdot}_{R^{\mathrm{aff}}[[x'_1,\dots,x'_d]]/R}$$

is a quasi-isomorphism. Hence the left inverse of this map (coming from the map $R^{\mathrm{aff}}[[x'_1,\dots,x'_d]] \to R^{\mathrm{aff}}$ given by sending $x'_i \to 0$)

$$\Omega^{\mathrm{cont},\cdot}_{R^{\mathrm{aff}}[[x'_1,\dots,x'_d]]/R} \to \Omega^{\cdot}_{R^{\mathrm{aff}}/R}$$

is also a quasi-isomorphism.

Combining this with (6.27) we see that the top map in (6.26) induces a quasi-isomorphism on relative De Rham complexes.

Using the proof of Proposition 6.3.1 we see that R^{aff} is a direct limit of finitely generated polynomial rings R_i over R. Thus we have $\Omega^{\cdot}_{R^{\mathrm{aff}}/R} = \operatorname{inj}\lim_i \Omega^{\cdot}_{R_i/R}$ and since it is well-known that $R \to \Omega_{R_i/R}$ is a quasi-isomorphism (the Poincare lemma) we obtain that θ_0 induces a quasi-isomorphism

$$R = \Omega^{\cdot}_{R/R} \to \Omega_{R^{\mathrm{aff}}/R}$$

Thus the right most map in (6.26) also induces a quasi-isomorphism on relative De Rham complexes. Therefore the left most one does as well. □

Example 6.6.2. As usual it is instructive to consider the case $R = k[x]$. As in Example 6.1.5 we have

$$R^{\mathrm{coord}} = k[x_0, x_1, \dots]_{x_1}$$

The one dimensional torus GL_1 acts with weight $-i$ on x_i (this follows form the fact that $\tilde{x} = \sum_i x_i t^i$ must be invariant). Hence

$$R^{\mathrm{aff}} = (k[x_0, x_1, \dots]_{x_1})^{\mathrm{GL}_1} = k[y_0, y_2, \dots]$$

where $y_i = (x_1)^{-i} x_i$. The map $R \to R^{\mathrm{aff}}$ is still given by $x \mapsto x_0 = y_0$ and the ideal $I = \ker(R^{\mathrm{aff}} \otimes R \to R^{\mathrm{aff}})$ is generated by $y_0 - x$.

We have

$$\Omega^{\cdot}_{R^{\mathrm{aff}}} = k[y_0, dy_0, y_2, dy_2, \dots]$$

where $\deg dy_i = 1$. Put

$$\Delta \overset{\mathrm{def}}{=} \Omega^{\cdot}_{R^{\mathrm{aff}}} \otimes R = k[y_0, dy_0, y_2, dy_2, \dots, x]$$

and thus

$$\Omega^{\cdot}_{R^{\mathrm{aff}}} \hat{\boxtimes} R = k[y_0, dy_0, y_2, dy_2, \dots, x]\hat{} \tag{6.28}$$

where the completion is graded completion with respect to the ideal $y_0 - x$. To prove directly that the homology of (6.28) is R it is sufficient to construct a *continuous* homotopy between the maps of DG-$k[x]$algebras

$$\phi_0 : \Delta \to \Delta : y_i \mapsto \begin{cases} x & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad dy_i \mapsto 0$$

and

$$\phi_1 : \Delta \to \Delta : y_i \mapsto y_i, dy_i \mapsto dy_i$$

(viewed as maps of complexes).

Introducing an auxiliary variable z, a functional homotopy between these two maps is given by the map of DG-$k[x]$-algebras

$$H : \Delta \mapsto \Delta \otimes k[z, dz]$$

$$H(y_i) = \begin{cases} z(y_0 - x) + x, & \text{if } i = 0 \\ zy_i & \text{otherwise} \end{cases}$$

and $H(dy_i) = d(H(y_i))$. By this we mean that $\phi_i = H\mid_{z=i,dz=0}$.

The following formula then yields a continuous homotopy between ϕ_0 and ϕ_1

$$h(\omega) = \int_{z=0}^{z=1} H(\omega) \tag{6.29}$$

The meaning of the right hand side of (6.29) is as follows. Write an element η of $\Delta \otimes k[z, dz]$ as $\eta_0(x, y, z) + \eta_1(x, y, z)dz$ where η_0 does not contain dz. Then

$$\int_{z=0}^{z=1} \eta \overset{\text{def}}{=} \int_{z=0}^{z=1} \eta_1(x, y, z)dz$$

6.7. **De Rham complexes of formal power series rings.** The following abstract result was used in the previous section.

Theorem 6.7.1. *Assume that T_0 is an R-algebra and that R is a k-algebra. Put $\hat{T} = T_0[[x_1, \ldots, x_n]]$. Then the canonical map*

$$\Omega^{\cdot}_{T_0/R} \to \Omega^{\cdot, cont}_{\hat{T}/R}$$

is a quasi-isomorphism of complexes of R-modules.

Proof. Put $T = T_0[t_1, \ldots, t_n]$. By Proposition 5.4.3 $\Omega^{\cdot,\text{cont}}_{\hat{T}/R}$ is the (graded) completion of $\Omega^{\cdot}_{T/R}$.

The latter is equal to $\Omega^{\cdot}_{T_0/R} \otimes_R \Omega^{\cdot}_{R[t_1,\ldots,t_n]/R}$ as graded commutative differential graded R-algebras. We view $\Omega^{\cdot}_{T/R}$ as a first quadrant double complex with the horizontal direction being given by $\Omega^{\cdot}_{R[t_1,\ldots,t_n]/R}$.

Hence it is sufficient to prove that for any T_0-module M the completion of

$$M \otimes_R \Omega^{\cdot}_{R[t_1,\ldots,t_n]/R} \tag{6.30}$$

has homology in degree zero and is acyclic elsewhere. By the Poincare lemma for polynomial rings this is true before completion. □

Now if we put $\deg t_i = \deg dt_i = 1$ then (6.30) is a graded complex. Hence for every n the part of degree n

$$(M \to M \otimes_R \Omega^{\cdot}_{R[t_1,\ldots,t_n]/R})_n \tag{6.31}$$

in (6.30) is exact (with M in degree zero). Now since (6.30) is a complex with positively graded components, its completion (augmented with M) is simply the product of the complexes (6.31). Hence it is exact also.

7. Reminder on DG-Lie and L_∞-algebras

7.1. **Coderivations.** Let V be a graded vector space and set $SV = \oplus_{n=0}^{\infty} S^n V$ considered as an augmented coalgebra such that

$$\Delta(v) = v \otimes 1 + 1 \otimes v$$
$$\epsilon(v) = 0$$

for $v \in V$. Fix $w = w_1 \cdots w_n$ where the w_i are homogeneous elements of V. Let $N = \{1, \ldots, n\}$. For $I \subset N$ put $w_I = \prod_{i \in I} w_i$. For a disjoint decomposition

$N = I_1 \cup \cdots \cup I_p$ we define $\epsilon(I_1, \ldots, I_n)$ as the sign which makes the following formula formally correct

$$w_{I_1} \cdots w_{I_p} = \epsilon(I_1, \ldots, I_p) w$$

A coderivation Q of degree one on SV is determined by its "Taylor coefficients" $(\partial^n Q)_{n\geq 0}$ which are the compositions

$$S^n V \xrightarrow{\text{inclusion}} SV \xrightarrow{Q} SV \xrightarrow{\text{projection}} V$$

Q can be computed from its Taylor coefficients by a kind of Leibniz rule. One has

$$Q(w) = \sum_{I \subset N} \epsilon(I, N-I)(\partial^{|I|} Q)(w_I) w_{N-I} \tag{7.1}$$

The Taylor coefficients of Q^2 are thus given by

$$(\partial^n Q^2)(w) = \sum_{I \subset N} \epsilon(I, N-I)(\partial^{n-|I|+1} Q)((\partial^{|I|} Q)(w_I) w_{N-I}) \tag{7.2}$$

We assume throughout that Q is compatible with the augmented structure. I.e. $Q(1) = 0$, or equivalently $\partial^0 Q = 0$. If $\partial^n Q = 0$ for $n > 1$ then (7.1) implies that Q is a *derivation* for the canonical algebra structure on SV.

7.2. **Coalgebra maps.** If V, W are graded vector spaces then an augmented coalgebra map of degree zero $\psi : SV \to SW$ is determined its "Taylor coefficients" $(\partial^n \psi)_{\geq 1}$ which are the compositions

$$S^n V \xrightarrow{\text{inclusion}} SV \xrightarrow{\psi} SW \xrightarrow{\text{projection}} W$$

ψ can be computed from its Taylor coefficients as follows.

$$\psi(w) = \sum_{N = I_1 \cup \cdots \cup I_p} \frac{1}{p!} \epsilon(I_1, \ldots, I_n)(\partial^{|I_1|}\psi)(w_{I_1}) \cdots (\partial^{|I_p|}\psi)(w_{I_p}) \tag{7.3}$$

Here $N = I_1 \cup \cdots \cup I_p$ is an ordered partition of N into p disjoint subsets (with p variable).

It follows from (7.3) that if $\partial^n \psi = 0$ for $n > 1$ then ψ is an *algebra homomorphism* $SV \to SW$.

Assume that SV and SW are equipped with a coderivation of degree one, denoted by Q. One may show that the condition

$$\psi \circ Q = Q \circ \psi$$

is equivalent to the corresponding "first order condition"

$$\partial^n(\psi \circ Q) = \partial^n(Q \circ \psi)$$

The latter condition maybe expanded as

$$\sum_{I \subset N} \epsilon(I, N-I)(\partial^{n-|I|+1}\psi)((\partial^{|I|} Q)(w_I) w_{N-I}) = \sum_{N = I_1 \cup \cdots \cup I_p} \frac{1}{p!} \epsilon(I_1, \ldots, I_n)(\partial^p Q)((\partial^{|I_1|}\psi)(w_{I_1}) \cdots (\partial^{|I_p|}\psi)(w_{I_p}) \tag{7.4}$$

For further reference we note that in case $\partial^i Q = 0$ for $i \neq 1, 2$ this formula specializes to

$$
\begin{aligned}
(7.5) \quad & \sum_{1\le i\le n} \epsilon(i, N-\{i\})(\partial^n\psi)((\partial^1 Q)(w_i)w_{N-\{i\}})+ \\
& \sum_{1\le i<j\le n} \epsilon(i,j,N-\{i,j\})(\partial^{n-1}\psi)((\partial^2 Q)(w_iw_j)w_{N-\{i,j\}}) = \\
& (\partial^1 Q)((\partial^n\psi)(w)) + \frac{1}{2}\sum_{N=I_1\cup I_2} \epsilon(I_1,I_2)(\partial^2 Q)((\partial^{|I_1|}\psi)(w_{I_1})(\partial^{|I_2|}\psi)(w_{I_2}))
\end{aligned}
$$

7.3. L_∞**-algebras and morphisms.**

Definition 7.3.1. An L_∞-structure on a vector space $\mathfrak{g}$ is a coderivation Q of degree one on $S(\mathfrak{g}[1])$ which has square zero.

One puts for $a \in \mathfrak{g}$

$$
\begin{aligned}
da &= -\partial^1 Q(a) \\
[a,b] &= (-1)^{|a|}\partial^2 Q(a,b)
\end{aligned}
\tag{7.6}
$$

(where $|a|$ is the degree of $a \in \mathfrak{g}$). It then follows from (7.2) that $d^2 = 0$ and that d is a derivation of degree one of $\mathfrak{g}$ with respect to the binary operation of degree zero $[-,-]$. If $\partial^i Q = 0$ for $i > 2$ then $\mathfrak{g}$ is a DG-Lie algebra. Conversely any DG-Lie algebra can be made into an L_∞-algebra by defining $\partial^1 Q$, $\partial^2 Q$ according (7.6) and by putting $\partial^i Q = 0$ for $i > 2$.

A morphism of L_∞-algebras $\mathfrak{g} \to \mathfrak{h}$ is by definition a coalgebra map of degree zero $S(\mathfrak{g}[1]) \to S(\mathfrak{h}[1])$ commuting with Q. It is customary to write $\psi_i = \partial^i\psi$ where ψ_i is considered as a map $\wedge^i\mathfrak{g} \to \mathfrak{h}$ of degree $1-n$. It follows from (7.4) that $d\psi_1 = \psi_1 d$. Hence ψ_1 defines a morphism of complexes.

7.4. **The topological case.** The above notions make sense in any symmetric monoidal category. We will use them in the case of filtered complete linear topological vector spaces.

7.5. **Twisting.** Assume that $\psi : \mathfrak{g} \to \mathfrak{h}$ is a L_∞-morphism between L_∞-algebras equipped with some type of topology and let $\omega \in \mathfrak{g}_1$ be a solution of the L_∞-Maurer-Cartan equation

$$\sum_{i\ge 1} \frac{1}{i!}(\partial^i Q)(\omega^i) = 0$$

in $\mathfrak{g}$. Here and below we assume that we are in a situation where all occurring series are convergent and standard series manipulations are allowed. This will be the case in the application in §9.1 where the series will in fact be finite.

Define Q_ω, ψ_ω and ω' by [37]

$$(\partial^i Q_\omega)(\gamma) = \sum_{j\ge 0} \frac{1}{j!}(\partial^{i+j} Q)(\omega^j\gamma) \qquad (\text{for } i>0) \tag{7.7}$$

$$(\partial^i \psi_\omega)(\gamma) = \sum_{j\ge 0} \frac{1}{j!}(\partial^{i+j} \psi)(\omega^j\gamma) \qquad (\text{for } i>0) \tag{7.8}$$

$$\omega' = \sum_{j\ge 1} \frac{1}{j!}(\partial^j\psi)(\omega^j) \tag{7.9}$$

for $\gamma \in S^i(\mathfrak{g}[1])$. Yekutieli shows in [37] that ω' is a solution of the Maurer-Cartan equation on $\mathfrak{h}$ and that furthermore $\mathfrak{g}$, $\mathfrak{h}$, when equipped with Q_ω, $Q_{\omega'}$ are again L_∞-algebras. Let us denote these by $\mathfrak{g}_\omega$ and $\mathfrak{h}_{\omega'}$. Yekutieli also shows that ψ_ω is an L_∞ map $\mathfrak{g}_\omega \to \mathfrak{h}_{\omega'}$. Variants of this principle occur at other places in the literature. See e.g. [33, Corollary 4.0.3][12, §2.4]. Let us see what the definition of Q_ω means in case $\mathfrak{g}$ is a DG-Lie algebra. In this case the L_∞-Maurer-Cartan equation translates into the usual Maurer-Cartan equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

Then

$$\begin{aligned}
(\partial^1 Q_\omega)(\gamma) &= (\partial^1 Q)(\gamma) + (\partial^2 Q)(\omega\gamma) \\
(\partial^2 Q_\omega)(\gamma) &= (\partial^2 Q)(\gamma) \\
(\partial^i Q_\omega)(\gamma) &= 0 \qquad (\text{for } i \geq 3)
\end{aligned}$$

Or translated into differentials and Lie brackets

$$\begin{aligned}
d_\omega &= d + [\omega, -] \\
[-,-]_\omega &= [-,-]
\end{aligned} \tag{7.10}$$

7.6. **Descent for L_∞-morphisms.** This is a somewhat abstract section. It is an explicitation of [26, §7.3.3] (in particular the last paragraph). The result will be used to descend a L_∞-morphism under a rational group action.

We now assume that $\mathfrak{g}$ is a DG-Lie algebra and $\mathfrak{s}$ is a set. We assume there is an "action" of $\mathfrak{s}$ on $\mathfrak{g}$ such that $v \in \mathfrak{s}$ acts by a derivation of degree -1 on $\mathfrak{g}$, denoted by i_v. Put $L_v = di_v + i_v d$. This is a derivation of $\mathfrak{g}$ of degree zero.

By the discussion in §7.1 there exist unique coderivations $\tilde{\imath}_v$ and $\tilde{L}_v$ on $S(\mathfrak{g}[1])$ such that $\partial^1\tilde{\imath}_v = j_v \overset{\text{def}}{=} -i_v$, $\partial^1\tilde{L}_v = L_v$ and $\partial^i\tilde{\imath}_v = \partial^i\tilde{L}_v = 0$ for $i \neq 1$ (the sign change on $\tilde{\imath}_v$ occurs because of the fact that i_v is an odd map $\mathfrak{g} \to \mathfrak{g}$ and $\partial^1\tilde{\imath}_v$ is the corresponding map $\mathfrak{g}[1] \to \mathfrak{g}[1]$).

Lemma 7.6.1. *One has*

$$\tilde{L}_v = [Q, \tilde{\imath}_v]$$

Proof. We know that $[Q, \tilde{\imath}_v]$ is a coderivation. Hence we have to compute $\partial^i[Q, \tilde{\imath}_v]$. Since $\tilde{\imath}_v$ maps $S^i(\mathfrak{g}[1])$ to $S^i(\mathfrak{g}[1])$ we have $\partial^i[Q, \tilde{\imath}_v] = \partial^i Q \circ \tilde{\imath}_v + \tilde{\imath}_v \circ \partial^i Q$. We need only to consider the cases $i = 1, 2$. Assume first $i = 2$. Then (using the fact that $\tilde{\imath}_v$ is also a derivation on $S(\mathfrak{g}[1])$, equal to j_v on $\mathfrak{g}[1]$) we compute for $a, b \in \mathfrak{g}$ (considered as elements of $\mathfrak{g}[1]$)

$$\begin{aligned}
(\partial^2 Q \circ \tilde{\imath}_v + \tilde{\imath}_v \circ \partial^2 Q)(a, b) &= \partial^2 Q(j_v a, b) + (-1)^{|a|-1}\partial^2 Q(a, j_v b) + j_v \circ \partial^2 Q(a, b) \\
&= (-1)^{|a|-1}[j_v a, b] - [a, j_v b] + (-1)^{|a|} j_v[a, b] \\
&= 0
\end{aligned}$$

(note that $|a|$, $|b|$ refer to the degrees of a, b in $\mathfrak{g}$).

Now assume $i = 1$. We have

$$(\partial^1 Q \circ \tilde{\imath}_v + \tilde{\imath}_v \circ \partial^1 Q)(a) = -dj_v(a) - j_v d(a) = L_v(a) = \tilde{L}_v(a) \quad \square$$

Put

$$\mathfrak{g}^{\mathfrak{s}} = \{X \in \mathfrak{g} \mid \forall v \in \mathfrak{s} : i_v X = L_v X = 0\} \tag{7.11}$$

we call $\mathfrak{g}^{\mathfrak{s}}$ the reduction of $\mathfrak{g}$ with respect to the $\mathfrak{s}$-action. It is clear that $\mathfrak{g}^{\mathfrak{s}}$ is a DG-Lie algebra as well.

Remark 7.6.2. It is perhaps useful to point out that whereas the notion of an $\mathfrak{s}$-action only depends on the graded structure of $\mathfrak{g}$, the construction of $\mathfrak{g}^{\mathfrak{s}}$ also depends on the differential.

Proposition 7.6.3. *Assume that ψ is an L_∞-morphism $\mathfrak{g}\rightarrow\mathfrak{h}$ between DG-Lie algebras equipped with a $\mathfrak{s}$-action as above. Assume that ψ commutes with the $\mathfrak{s}$ action in the sense that for all $v\in\mathfrak{s}$*

$$\forall v : [\tilde{\imath}_v,\psi]=0$$

(where as above $\tilde{\imath}_v$ stands for the induced coderivations on $S(\mathfrak{g}[1])$ and $S(\mathfrak{h}[1])$). Then ψ descends to an L_∞-morphism $\psi^{\mathfrak{s}}:\mathfrak{g}^{\mathfrak{s}}\rightarrow\mathfrak{h}^{\mathfrak{s}}$.

Proof. By (7.11) we have to show that the restrictions of $\tilde{\imath}_v\circ\partial^i\psi$ and $\tilde{L}_v\circ\partial^i\psi$ to $S^i(\mathfrak{g}^{\mathfrak{s}}[1])$ are zero. Note that since $\tilde{L}_v=[Q,\tilde{\imath}_v]$ (by Lemma 7.6.1, the fact that $\tilde{\imath}_v$ commutes with ψ implies that $\tilde{L}_v$ commutes with ψ as well.

$$\begin{aligned}\tilde{\imath}_v\circ\partial^i\psi&=\partial^i(\tilde{\imath}_v\circ\psi)=\partial^i(\psi\circ\tilde{\imath}_v)\\ \tilde{L}_v\circ\partial^i\psi&=\partial^i(\tilde{L}_v\circ\psi)=\partial^i(\psi\circ\tilde{L}_v)\end{aligned}$$

Since $\tilde{\imath}_v$ and $\tilde{L}_v$ are zero on $S(\mathfrak{g}^{\mathfrak{s}}[1])$ this implies the desired result. □

7.7. **Compatibility with twisting.** Assume that $\mathfrak{g}$, $\mathfrak{h}$ are topological L_∞-algebras and ψ is an L_∞-morphism $\mathfrak{g}\rightarrow\mathfrak{h}$. We make the same assumptions as in §7.5 with regard to convergence of series. Our aim to understand the behavior of $\mathfrak{s}$-actions under twisting.

Proposition 7.7.1. *Assume that $\mathfrak{g}$ and $\mathfrak{h}$ are equipped with a $\mathfrak{s}$-action and assume that ψ commutes with this action (as in Proposition 7.6.3). Let $\omega\in\mathfrak{g}_1$ be a solution to the Maurer Cartan equation. Since twisting does not change the Lie bracket (see (7.10)), $\mathfrak{s}$ acts on $\mathfrak{g}_\omega$ and $\mathfrak{h}_\omega$ as well.*

Assume that for $i\geq 2$ and all $v\in\mathfrak{s}$, $\gamma\in S^{i-1}(\mathfrak{g}[1])$ we have

$$(\partial^i\psi)(i_v\omega\cdot\gamma)=0 \tag{7.12}$$

Then ψ_ω is compatible with the $\mathfrak{s}$-action on $\mathfrak{g}_\omega$ and $\mathfrak{h}_\omega$.

Proof. $[\tilde{\imath}_v,\psi_\omega]$ is a "ψ_ω-coderivation" which is the dual notion of a "ϕ-derivation" for a map of algebras $\phi:A\rightarrow B$. One verifies that in order to prove $[\tilde{\imath}_v,\psi_\omega]=0$ it is sufficient to show that $\partial^i[\tilde{\imath}_v,\psi_\omega]=\tilde{\imath}_v\circ\partial^i\psi_\omega-\partial^i\psi_\omega\circ\tilde{\imath}_v=0$.

We have for $\gamma\in S^i(\mathfrak{g}[1])$, $i>0$

$$(\partial^i\psi_\omega\circ\tilde{\imath}_v)(\gamma)=\sum_{j\geq 0}\frac{1}{j!}(\partial^{i+j}\psi)(\omega^j\cdot\tilde{\imath}_v\gamma) \tag{7.13}$$

and

(7.14)

$$\begin{aligned}(\tilde{\imath}_v\circ\partial^i\psi_\omega)(\gamma)&=\sum_{j\geq 0}\frac{1}{j!}(\partial^{i+j}\psi)(\tilde{\imath}_v(\omega^j\gamma))\\ &=\sum_{j\geq 1}\left(\frac{1}{(j-1)!}(\partial^{i+j}\psi)(j_v(\omega)\cdot\omega^{j-1}\gamma)\right)+\left(\frac{1}{j!}(\partial^{i+j}\psi)(\omega^j\cdot\tilde{\imath}_v\gamma)\right)\end{aligned}$$

(recall that ω has degree zero in $\mathfrak{g}[1]$). The difference between (7.13) and (7.14) is a linear combination of terms of the form (7.12) and hence it is zero. □

8. Poly-differential operators and poly-vector fields revisited

In this section we remind the reader about some facts on poly-differential operators and poly-vector fields. These notions were already introduced in the introduction but for the convenience of the reader we repeat some definitions. From now on we assume $k = \mathbb{C}$.

8.1. **General definitions.** Let R be a finite adic k-algebra. We view R as a $R^{\otimes n}$ module through the diagonal action. We put

$$D^{\text{poly},n}(R) = \text{Diff}_{R^{\otimes n}}(R^{\otimes n}, R)$$

where "Diff" stands for differential operators.[1] We also write $D^{\text{poly},n}(X) = D^{\text{poly},n}(R)$ if $X = \operatorname{Spc} R$. If X is a finite adic scheme then we define $\mathcal{D}_X^{\text{poly},n}$ by gluing from the affine case.

We may view the elements of $D^{\text{poly},n}(R)$ as the set of multilinear maps $R^{\otimes n} \to R$ which are differential operators in each of their arguments. With this interpretation it is clear that $D^{\text{poly},\cdot}(R)$ is a DG-Lie subalgebra of $\mathbf{C}^{\cdot}(R)$, the Hochschild complex of R. In particular it is a DG-Lie subalgebra.

We say that R is formally of finite type [35] if R has a finitely generated ideal of definition I such that R/I is finitely generated. This definition is clearly independent of I. If R is formally of finite type then let us say that R is formally smooth if $\Omega_R^{1,\text{cont}}$ is projective. In that case following Yekutieli's argument in [36] we see that the map

$$D^{\text{poly},\cdot}(R) \to \mathbf{C}^{\cdot}(R)$$

is a quasi-isomorphism.

For $p \geq 0$ let $F^p D^{\text{poly},n}(R)$ be the differential operators of degree $\leq p$. It is then easy to see that if R is formally of finite type then $F^p D^{\text{poly},n}(R)$ is a finite R-module. In that case we will view $D^{\text{poly},n}(R)$ as a filtered complete linear topological vector space.

Similarly put

$$T^{\text{poly},n}(R) = \text{Der}_{R^{\otimes n}}(R^{\otimes n}, R)^{S^n}$$

The righthand side describes the set of poly-derivations which are anti-symmetric in their arguments.[2]

If R is formally of finite type then $T^{\text{poly},n}(R)$ is a finite R-module. In that case we view $T^{\text{poly},n}(R)$ as a filtered complete linear topological vector space with filtration concentrated in degree n.

Convention-Warning 8.1.1. *In this section and the next almost all our objects will be considered as being (naturally) filtered. This has serious implications for the meaning of completions and completed tensor products. See §5.2.*

---

[1] Note that since differential operators are continuous with respect to any adic topology, we don't have to worry about continuity.

[2] In §5.4 the notation $\text{Der}^n(R, R)$ was used. The current notation is more convenient for this section

If R formally of finite type and formally smooth then there is an isomorphism

$$\wedge^n_R T^{\text{poly},1}(R) \to T^{\text{poly},n}(R) : \gamma_1 \wedge \cdots \wedge \gamma_n \mapsto \sum_{\sigma \in S_n} (-1)^\sigma \gamma_{\sigma(1)} \otimes \cdots \otimes \gamma_{\sigma(n)}$$

where $\gamma_i \in T^{\text{poly}}(R) = \operatorname{Der}_k(R,R)$ and $\gamma_1 \otimes \cdots \otimes \gamma_n$ acts on $R^{\otimes n}$ via

$$(\gamma_1 \otimes \cdots \otimes \gamma_n)(r_1 \otimes \cdots \otimes r_n) = \gamma_1(r_1) \cdots \gamma_n(r_n)$$

8.2. **The formal case.** In this section we consider $R = k[[t_1, \ldots, t_d]]$. Write ∂_i for $\partial/\partial t_i$. In this case we can give very concrete descriptions of $T^{\text{poly},\cdot}(R)$ and $D^{\text{poly},\cdot}(R)$. First we have

$$\begin{aligned} T^{\text{poly},1}(R) &= R\partial_1 \oplus \cdots \oplus R\partial_d \\ D^{\text{poly},1}(R) &= R[\partial_1, \cdots, \partial_d] \end{aligned} \tag{8.1}$$

and then

$$\begin{aligned} T^{\text{poly},n}(R) &= \wedge^n_R T^{\text{poly},n} \\ D^{\text{poly},n}(R) &= \otimes^n_R D^{\text{poly},} \end{aligned} \tag{8.2}$$

These descriptions reflect the algebra structure on $T^{\text{poly},\cdot}(R)$ and $D^{\text{poly},\cdot}(R)$. It is also easy to get the Lie algebra structure on $T^{\text{poly},\cdot}(R)$ using the fact that the product satisfies the Leibniz property with respect to the Lie bracket. In $D^{\text{poly},\cdot}(R)$ this Leibniz property holds only up to homotopy and therefore the situation is much more complicated.

Kontsevich (over the reals) constructs in [26] an L_∞-quasi-isomorphism

$$\mathcal{U} : T^{\text{poly},\cdot}(R)[1] \to D^{\text{poly},\cdot}(R)[1] \tag{8.3}$$

If we write $\mathcal{U}_i = \partial^i \mathcal{U}$ then $\mathcal{U}_1$ is given by the HKR formula

$$\mathcal{U}_1(\partial_{i_1} \wedge \cdots \wedge \partial_{i_p}) = \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma \partial_{i_{\sigma(1)}} \otimes \cdots \otimes \partial_{i_{\sigma(p)}} \tag{8.4}$$

The higher $\mathcal{U}_n$ are matrices of differential operators when expressed in the natural R-bases of $T^{\text{poly},n}(R)$ and $D^{\text{poly},n}(R)$ obtained from (8.1)(8.2).

The quasi-isomorphism constructed by Kontsevich has two supplementary properties which are crucial for its extension to the non-formal case.

(P4) $\mathcal{U}_q(\gamma_1 \cdots \gamma_q) = 0$ for $q \geq 2$ and $\gamma_1, \ldots, \gamma_q \in T^{\text{poly},1}(R)$.
(P5) $\mathcal{U}_q(\gamma\alpha) = 0$ for $q \geq 2$ and $\gamma \in \mathfrak{gl}_d(k) \subset T^{\text{poly},1}(R)$.

Remark 8.2.1. In [32] Tamarkin constructs an L_∞-quasi-isomorphism like (8.3) over the rationals. Halbout has informed me that the methods in [17] show that Tarmarkin's quasi-isomorphism may be defined in such a way that it also satisfies (P4) and (P5). Using this one may replace the complex numbers by the rational numbers in this paper.

Remark 8.2.2. Another property which is usually being regarded as essential for globalization is the fact that the $\mathcal{U}_q$ are $\operatorname{GL}_d(k)$ equivariant (condition (P3) in [26]). We will not explicitly use this condition below. The explanation for this is that (P3) almost follows from (P5). To be more precise let (P3') be the condition that $\mathcal{U}_q$ is $\mathfrak{gl}_d(k)$ equivariant. I.e.

(P3') $[\gamma, \mathcal{U}_q(\alpha_1 \cdots \alpha_q)] = \sum_j \mathcal{U}_q(\alpha_1 \cdots [\gamma, \alpha_j] \cdots \alpha_q)$

for $\gamma \in \mathfrak{gl}_d(k)$, $\alpha_i \in T^{\text{poly},\cdot}(R)[1]$.

Then we have $(P5) \Rightarrow (P3')$. This easily follows from (7.5). In sufficiently nice situations (P3) and (P3') are of course equivalent.

9. Global formality

9.1. **Lifting to coordinate spaces.** It is easy to define relative poly-differential operators with respect to a graded commutative base ring. Assume that $A \to B$ is a morphism of graded commutative algebras and let M be a graded B-module. We define $D_A^{\mathrm{poly},n}(B,M)$ as the set of multilinear maps $B \otimes_A \cdots \otimes_A B \to M$ (n copies of B) which are relative B/A differential operators in each of their arguments and which are finite sums of homogeneous maps.

We will use the following routine lemma to manipulate such relative poly-differential operators.

Lemma 9.1.1. *Assume that A is a graded commutative DG-algebra and let S be a finitely generated smooth k-algebra. Let I be a finitely generated ideal in $A_0 \otimes S$ (A_0 is the part of degree zero of A). Then the obvious map of DG-Lie algebras*

$$D^{poly,\cdot}(S) \to D_A^{poly,\cdot}(A \hat{\boxtimes} S) \tag{9.1}$$

(all completions are I-adic completions of filtered topological $A_0 \otimes S$-modules) extends to an isomorphism of double complexes of filtered complete linear topological vector spaces

$$A \hat{\boxtimes} D^{poly,\cdot}(S) \to D_A^{poly,\cdot}(A \hat{\boxtimes} S) \tag{9.2}$$

if we define the vertical differentials as the Hochschild differentials (considering $A \hat{\boxtimes} S$ as a graded ring) and the horizontal differentials on the right as $[d_A \otimes 1, -]$ and on the left as $d_A \otimes 1$.

Proof. It is easy to see that the differentials are as indicated. So we only have to show that (9.2) is an isomorphism.

Since differential operators are always continuous with respect to the I-adic topology we have

$$F^p D_A^{\mathrm{poly},n}(A \hat{\boxtimes} S) = F^p D_A^{\mathrm{poly},n}(A \otimes S, A \hat{\boxtimes} S) = F^p D^{\mathrm{poly},n}(S, A \hat{\boxtimes} S)$$

where as in §8.1 $F^{\cdot}$ denotes the filtration by degree of differential operators.

From the standard theory of differential operators it follows that there exists a *finitely generated projective* $S^{\otimes n}$ module J_p such that

$$F^p D^{\mathrm{poly},n}(S,-) = \mathrm{Hom}_{S^{\otimes n}}(J_p, -)$$

Hence

$$\begin{aligned} F^p D^{\mathrm{poly},n}(S, A \hat{\boxtimes} S) &= \mathrm{Hom}_{S^{\otimes n}}(J_p, A \hat{\boxtimes} S) \\ &= \mathrm{Hom}_{S^{\otimes n}}(J_p, \operatorname{proj\,lim}_m (A \otimes S)/I^m) \\ &= \operatorname{proj\,lim}_m \mathrm{Hom}_{S^{\otimes n}}(J_p, (A \otimes S)/I^m) \\ &= \operatorname{proj\,lim}_m (A \otimes \mathrm{Hom}_{S^{\otimes n}}(J_p, S))/I^m \\ &= A \hat{\boxtimes} F^p D^{\mathrm{poly},n}(S) \end{aligned}$$

In the third line we have used the fact that J_p is finitely generated projective. This allows us to replace J_p by $S^{\otimes n}$. □

Assume now that R is smooth of dimension d. Using the previous lemma first with $A = \Omega^\cdot_{R^{\text{coord}}}$, $S = R$ and $I = \ker(R^{\text{coord}} \otimes R \to R^{\text{coord}})$ and then with $A = \Omega^\cdot_{R^{\text{coord}}}$, $S = k[t_1, \dots, t_d]$, $I = (t_1, \dots, t_d)$ we have the following string of maps between of filtered complete linear topological DG-Lie algebras.

$$
\begin{aligned}
D^{\text{poly},\cdot}(R) &\to \Omega^\cdot_{R^{\text{coord}}} \hat{\boxtimes} D^{\text{poly},\cdot}(R) \\
&\cong D^{\text{poly},\cdot}_{\Omega^\cdot_{R^{\text{coord}}}}(\Omega^\cdot_{R^{\text{coord}}} \hat{\boxtimes} R) \\
&\cong D^{\text{poly},\cdot}_{\Omega^\cdot_{R^{\text{coord}}}}(\Omega^\cdot_{R^{\text{coord}}}[[t_1, \dots, t_d]]) \\
&\cong \Omega^\cdot_{R^{\text{coord}}} \hat{\boxtimes} D^{\text{poly},\cdot}(k[t_1, \dots, t_d]) \\
&\cong \Omega^\cdot_{R^{\text{coord}}} \hat{\otimes} D^{\text{poly},\cdot}(k[[t_1, \dots, t_d]])
\end{aligned}
\tag{9.3}
$$

In the third line we have used the isomorphism (6.22).

In §6.5 we have seen that the differential on $\Omega^{\cdot,\text{cont}}_{R^{\text{coord}}\hat{\boxtimes}R/R} = \Omega^\cdot_{R^{\text{coord}}} \hat{\boxtimes} R$ induces the differential $d + \omega_{MC}$ on $\Omega^\cdot_{R^{\text{coord}}}[[t_1, \dots, t_d]]$ with

$$
\omega_{MC} \in \Omega^1_{R^{\text{coord}}} \hat{\otimes} \text{Der}_k(k[[t_1, \dots, t_d]]) = \Omega^1_{R^{\text{coord}}} \hat{\otimes} T^{\text{poly},1}(k[[t_1, \dots, t_d]]) \subset \Omega^1_{R^{\text{coord}}} \hat{\otimes} D^{\text{poly},1}(k[[t_1, \dots, t_d]])
\tag{9.4}
$$

This then induces the differential $[d, -] + [\omega_{MC}, -] + d_{\text{Hoch}}$ on $D^{\text{poly},\cdot}_{\Omega^\cdot_{R^{\text{coord}}}}(\Omega^\cdot_{R^{\text{coord}}}[[t_1, \dots, t_d]])$. It is easy to see that this induces the differential $d \otimes 1 + [\omega_{MC}, -] + d_{\text{Hoch}}$ on the graded Lie algebra $\Omega^\cdot_{R^{\text{coord}}} \hat{\otimes} D^{\text{poly},\cdot}(k[[t_1, \dots, t_d]])$.

As for differential operators, we can define relative poly-vector fields and there is an obvious analog of lemma 9.1.1. We obtain maps of graded vector spaces which we may employ to get the following string of maps between filtered complete linear topological DG-Lie algebras.

$$
\begin{aligned}
T^{\text{poly},\cdot}(R) &\to \Omega^\cdot_{R^{\text{coord}}} \hat{\boxtimes} T^{\text{poly},\cdot}(R) \\
&\to T^{\text{poly},\cdot}_{\Omega^\cdot_{R^{\text{coord}}}}(\Omega^\cdot_{R^{\text{coord}}} \hat{\boxtimes} R) \\
&\cong T^{\text{poly},\cdot}_{\Omega^\cdot_{R^{\text{coord}}}}(\Omega^\cdot_{R^{\text{coord}}}[[t_1, \dots, t_d]]) \\
&\cong \Omega^\cdot_{R^{\text{coord}}} \hat{\boxtimes} T^{\text{poly},\cdot}(k[t_1, \dots, t_d]) \\
&\cong \Omega^\cdot_{R^{\text{coord}}} \hat{\otimes} T^{\text{poly},\cdot}(k[[t_1, \dots, t_d]])
\end{aligned}
\tag{9.5}
$$

On the first Lie algebra this differential is trivial and on the last it is $d \otimes 1 + [\omega_{MC}, -]$.

Now consider the following two DG-Lie algebra

$$
\mathfrak{t} = \Omega^\cdot_{R^{\text{coord}}} \hat{\otimes} T^{\text{poly},\cdot}_{k[[t_1, \cdots, t_d]]}(k[[t_1, \dots, t_d]])
\tag{9.6}
$$

$$
\mathfrak{d} = \Omega^\cdot_{R^{\text{coord}}} \hat{\otimes} D^{\text{poly},\cdot}_{k[[t_1, \dots, t_d]]}(k[[t_1, \dots, t_d]])
\tag{9.7}
$$

where the DG-Lie algebra structures are obtained by linearly extending the ones on the second component. Looking at the second line and the last line in (9.3) and (9.5) and the description of the differentials we obtain by (7.10)

$$
\begin{aligned}
\mathfrak{t}_{\omega_{MC}} &= \Omega^\cdot_{R^{\text{coord}}} \hat{\boxtimes} T^{\text{poly},\cdot}(R) \\
\mathfrak{d}_{\omega_{MC}} &= \Omega^\cdot_{R^{\text{coord}}} \hat{\boxtimes} D^{\text{poly},\cdot}(R)
\end{aligned}
\tag{9.8}
$$

We now extend the L_∞ quasi-isomorphism

$$\mathcal{U} : T^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]]) \to D^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]])$$

$\Omega^{\cdot}_{R^{\mathrm{coord}}}$-linearly to an L_∞-map $\bar{\mathcal{U}} : \mathfrak{t} \to \mathfrak{d}$.

Property (P4) together with (9.4) implies that if we plug ω_{MC} into (7.9) we obtain $\omega'_{MC} = \omega_{MC}$ (taking into account that the right most inclusion in (9.4) is simply $\bar{\mathcal{U}}_1$).

Lemma 9.1.2. *Assume that* $\gamma \in \Omega^a_{R^{\mathrm{coord}}} \hat{\boxtimes} T^{\mathrm{poly},b}(R)$. *Then the sum in* (7.8) *is finite.*

Proof. This follows by degree considerations. Indeed $\bar{\mathcal{U}}_i$ is obtained by extension of the map of degree zero

$$\mathcal{U}_i : S^i(T^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]])[2]) \to D^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]])[2]$$

It follows that $\bar{\mathcal{U}}_i$ considered as a map

$$S^i(\Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\otimes} T^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]])) \to \Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\otimes} D^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]])$$

has bidegree $(0, 2-2i)$. One the other hand it follows from (9.4) that ω_{MC} has bidegree $(1,1)$. Assume that γ has bidegree (a,b). Then the j'th term in (7.8) has bidegree $(a,b)+(j,j)+(0,2-2(i+j)) = (a+j, b-j+2-2i)$. Since for $j \gg 0$ we have $b-j+2-2i < 0$ it follows that the sum in (7.8) is indeed finite (for a given bihomogeneous γ). □

Hence we obtain that

$$\bar{\mathcal{U}}_{\omega_{MC}} : \mathfrak{t}_{\omega_{MC}} \to \mathfrak{d}_{\omega_{MC}} \tag{9.9}$$

is defined. From the formula (7.8) it follows that $\bar{\mathcal{U}}_{\omega_{MC}}$ is still $\Omega^{\cdot}_{R^{\mathrm{coord}}}$-linear.

9.2. **Descent.** If we let $v \in \mathfrak{g} = \mathrm{Der}_k(k[[t_1,\ldots,t_d]])$ act by $i_{\bar{v}}$ on $\Omega^{\cdot}_{R^{\mathrm{coord}},\cdot}$ (as in Lemma 6.5.1) then this defines a $\mathfrak{g}$-action on $\Omega^{\cdot}_{R^{\mathrm{coord}}}$ in the sense of §7.6. We define a corresponding $\mathfrak{g}$-action on $\mathfrak{t}$ and $\mathfrak{d}$ by linearly extension.

Lemma 9.2.1. *Put* $\mathfrak{s} = \mathrm{Lie}\,\mathrm{GL}_d = \mathfrak{gl}_d \subset \mathfrak{g}$. *The* L_∞*-morphism* (9.9) *descends to an* L_∞*-morphism*

$$(\bar{\mathcal{U}}_{\omega_{MC}})^{\mathfrak{s}} : (\mathfrak{t}_{\omega_{MC}})^{\mathfrak{s}} \to (\mathfrak{d}_{\omega_{MC}})^{\mathfrak{s}}$$

(where $(-)^{\mathfrak{s}}$ *is defined by* (7.11)*).*

Proof. According to Proposition 7.6.3 we need to show that the $\mathfrak{s}$ action on $\mathfrak{t}_{\omega_{MC}}$ and $\mathfrak{d}_{\omega_{MC}}$ is compatible with $\bar{\mathcal{U}}_{\omega_{MC}}$. By Proposition 7.7.1 it is sufficient to prove the following two statements

(1) The $\mathfrak{s}$ action on $\mathfrak{t}$ and $\mathfrak{d}$ is compatible with $\bar{\mathcal{U}}$.
(2) For $j \geq 2$ the condition

$$\bar{\mathcal{U}}_j(i_{\bar{v}}\omega \cdot \gamma) = 0$$

is satisfied.

Since $\bar{\mathcal{U}}$ is a base extension of $\mathcal{U}$, it is easy to see that it commutes with the action of $i_{\bar{v}}$, $v \in \mathfrak{s}$ (in the sense of Proposition 7.6.3). This proves (1).

Using Lemma 6.5.1 and expanding γ as a $\Omega^{\cdot}_{R^{\mathrm{coord}}}$-linear combination of elements γ' of $S(\mathfrak{t}[1])$ it follows that for (2) it is sufficient to prove that

$$\mathcal{U}_j(v \cdot \gamma') = 0$$

But this is precisely property (P5). □

Lemma 9.2.2. *The formulas* (9.8) *descend to isomorphisms of filtered complete linear topological DG-Lie algebras*

$$(\mathfrak{d}\omega_{MC})^{\mathfrak{s}} \cong \Omega^{\cdot}_{R^{\mathrm{aff}}} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R)$$

$$(\mathfrak{t}_{\omega_{MC}})^{\mathfrak{s}} \cong \Omega^{\cdot}_{R^{\mathrm{aff}}} \hat{\boxtimes} T^{\mathrm{poly},\cdot}(R)$$

where the completion is computed with respect to the ideal $\ker(R^{\mathrm{aff}} \otimes R \to R^{\mathrm{aff}})$.

Proof. We will concentrate ourselves on $\mathfrak{d}$. The case of $\mathfrak{t}$ is similar. By (9.8) we have

$$(\mathfrak{d}_{\omega_{MC}})^{\mathfrak{s}} = \left(\Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R)\right)^{\mathfrak{s}}$$

Thus we have to understand how the $\mathfrak{s}$-action on $\Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\otimes} D^{\mathrm{poly},\cdot}(k[[t_1,\dots,t_d]])$ is transported under the isomorphisms of (9.3) to a $\mathfrak{s}$-action on $\Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R)$. We will do this for the $\mathfrak{g}$-action. Since $\mathfrak{s} \subset \mathfrak{g}$ this does what we want.

We claim that the transported $\mathfrak{g}$-action is just the extension of the $\mathfrak{g}$-action on $\Omega^{\cdot}_{R^{\mathrm{coord}}}$. To prove this we observe that the isomorphism (6.12)

$$\Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\boxtimes} R \to \Omega^{\cdot}_{R^{\mathrm{coord}}}[[t_1,\dots,t_d]] : \omega \otimes f \mapsto \omega \tilde{f}$$

commutes with the $\mathfrak{g}$-actions on both sides (obtained from extending the $\mathfrak{g}$ action on $\Omega^{\cdot}_{R^{\mathrm{coord}}}$). To see this note that since $i_{\bar{v}}$ has degree -1, $i_{\bar{v}}$ is $R^{\mathrm{coord}} \hat{\boxtimes} R$-linear on the left and $R^{\mathrm{coord}}[[t_1,\dots,t_d]]$-linear on the right.

If $v \in \mathfrak{g}$ then $i_{\bar{v}}$ acts as a derivation on $\Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\boxtimes} R$ and $\Omega^{\cdot}_{R^{\mathrm{coord}}}[[t_1,\dots,t_d]]$ which preserves $\Omega^{\cdot}_{R^{\mathrm{coord}}}$ and this implies that $[i_{\bar{v}}, -]$ acts on

$$D^{\mathrm{poly},\cdot}_{\Omega^{\cdot}_{R^{\mathrm{coord}}}}(\Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\boxtimes} R) \tag{9.10}$$

and

$$D^{\mathrm{poly},\cdot}_{\Omega^{\cdot}_{R^{\mathrm{coord}}}}(\Omega^{\cdot}_{R^{\mathrm{coord}}}[[t_1,\dots,t_d]]) \tag{9.11}$$

and of course these actions are compatible with the isomorphism between (9.10) and (9.11).

It follows that we obtain compatible $\mathfrak{g}$-actions on

$$\Omega_{R^{\mathrm{coord}},\cdot} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R)$$

and

$$\Omega_{R^{\mathrm{coord}},\cdot} \hat{\otimes} D^{\mathrm{poly},\cdot}(k[[t_1,\dots,t_d]])$$

and it is easy to see that these are obtained from the $\mathfrak{g}$-actions on $\Omega_{R^{\mathrm{coord}},\cdot}$.

It remains to show

$$(\Omega_{R^{\mathrm{coord}},\cdot} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R))^{\mathfrak{s}} = \Omega_{R^{\mathrm{aff}},\cdot} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R)$$

This meant to be an isomorphism of filtered objects so we first consider $(\Omega_{R^{\mathrm{coord}},\cdot} \hat{\boxtimes} F^p D^{\mathrm{poly},\cdot}(R))^{\mathfrak{s}}$ which can be rewritten as

$$((\Omega_{R^{\mathrm{coord}},\cdot} \hat{\boxtimes} R) \otimes_R F^p D^{\mathrm{poly},\cdot}(R))^{\mathfrak{s}} = (\Omega_{R^{\mathrm{coord}},\cdot} \hat{\boxtimes} R)^{\mathfrak{s}} \otimes_R D^{\mathrm{poly},\cdot}(R)$$

since the $F^p D^{\mathrm{poly},\cdot}(R)$ are finitely generated projective R-modules.

So now we have to show

$$(\Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\boxtimes} R)^{\mathfrak{s}} = \Omega^{\cdot}_{R^{\mathrm{aff}}} \hat{\boxtimes} R$$

We use the easily proved fact that $\Omega^{\cdot}_{R^{\mathrm{coord}}} \hat{\boxtimes} R = \Omega^{\cdot,\mathrm{cont}}_{R^{\mathrm{coord}}\hat{\boxtimes} R/R}$. Let I be the kernel of $R^{\mathrm{coord}} \otimes R \to R^{\mathrm{coord}}$. Using Proposition 5.4.1 and Lemma 9.2.3 below we have

$$\begin{aligned}\left(\Omega^{m,\mathrm{cont}}_{R^{\mathrm{coord}}\hat{\boxtimes} R/R}\right)^{\mathfrak{s}} &= \left(\operatorname{proj}\lim_n \Omega^m_{(R^{\mathrm{coord}}\otimes R)/I^n/R}\right)^{\mathfrak{s}}\\ &= \operatorname{proj}\lim_n \left(\Omega^m_{(R^{\mathrm{coord}}\otimes R)/I^n/R}\right)^{\mathfrak{s}}\\ &= \operatorname{proj}\lim_n \left(\Omega^m_{((R^{\mathrm{coord}}\otimes R)/I^n)^{\mathrm{GL}_d}/R}\right)\end{aligned}$$

Now let J be the kernel of $R^{\mathrm{aff}} \otimes R \to R^{\mathrm{aff}}$. From the fact that GL_d acts freely on R^{coord} (Proposition 6.2.2) and its invariants are defined as R^{aff} we easily deduce that $((R^{\mathrm{coord}} \otimes R)/I^n)^{\mathrm{GL}_d} = (R^{\mathrm{aff}} \otimes R)/J^n$. Hence

$$\begin{aligned}\left(\Omega^{m,\mathrm{cont}}_{R^{\mathrm{coord}}\hat{\boxtimes} R/R}\right)^{\mathfrak{s}} &= \operatorname{proj}\lim_n \left(\Omega^m_{((R^{\mathrm{aff}}\otimes R)/J^n)/R}\right)\\ &= \Omega^{m,\mathrm{cont}}_{(R^{\mathrm{aff}}\hat{\boxtimes} R)/R}\end{aligned}$$

where we have used Proposition 5.4.1 once again. □

The following lemma was used.

Lemma 9.2.3. *Let S be a connected reductive algebraic group over k with Lie algebra $\mathfrak{s}$ acting rationally, R-linearly and freely on a R-algebra T. For $v \in \mathfrak{s}$ we denote by i_v the derivation of degree -1 on $\Omega^{\cdot}_{T/R}$ which is the contraction with the derivation corresponding to v. Then*

$$\begin{aligned}\Omega^{\cdot}_{T^S/R} &= \{\omega \in \Omega^{\cdot}_{T/R} \mid \forall v \in \mathfrak{s} : i_v(\omega) = i_v(d\omega) = 0\}\\ &= (\Omega^{\cdot}_{T/R})^{\mathfrak{s}}\end{aligned}$$

Proof. We use a fragment of the Cartan model for equivariant cohomology. Let $(e_j)_j$ be a basis for $\mathfrak{s}$ and let $(e_j^*)_j$ be the corresponding dual basis. Since S acts freely on T, T/T^S is smooth. We obtain an exact sequence

$$0 \to \Omega^1_{T^S/T} \otimes_{T^G} T \to \Omega^1_{T/R} \xrightarrow{\sum_j i_{e_j}\otimes e_j^*} T \otimes \mathfrak{s}^* \to 0$$

This sequence is split and hence we may transform into a Koszul type long exact sequence

$$0 \to \Omega^i_{T^S/R} \otimes_{T^S} T \to \Omega^i_{T/R} \xrightarrow{\delta} \Omega^{i-1}_{T/R} \otimes \mathfrak{s}^* \xrightarrow{\delta} \Omega^{i-2}_{T/R} \otimes S^2\mathfrak{s}^* \xrightarrow{\delta} \cdots \tag{9.12}$$

where

$$\delta(\omega \otimes f) = \sum_j i_{e_j}(\omega) \otimes e_j^* f$$

Taking invariants we obtain in particular

$$\Omega^{\cdot}_{T^S/R} = \{\omega \in (\Omega^{\cdot}_{T/R})^S \mid \forall v \in \mathfrak{s} : i_v(\omega) = 0\}$$

The differentiated S action of $\Omega^{\cdot}_T$ is given by $L_v = di_v + i_v d$. Hence we obtain

$$\Omega^{\cdot}_{T^S/R} = \{\omega \in (\Omega^{\cdot}_{T/R}) \mid \forall v \in \mathfrak{s} : i_v(\omega) = L_v(\omega) = 0\}$$

which yields the desired result. □

9.3. **Quasi-isomorphisms.**

Theorem 9.3.1. *The canonical maps*

$$T^{poly,\cdot}(R) \to \Omega_{R^{aff}} \hat{\boxtimes} T^{poly,\cdot}(R)$$
$$D^{poly,\cdot}(R) \to \Omega_{R^{aff}} \hat{\boxtimes} D^{poly,\cdot}(R)$$

obtained by linearly extending $R^{\mathrm{aff}} \to \Omega^{\cdot}_{R^{\mathrm{aff}}}$*, are filtered quasi-isomorphisms.*

Proof. By acyclicity (Theorem 6.6.1) we have a quasi-isomorphism

$$R \to \Omega^{\cdot,\mathrm{cont}}_{R^{\mathrm{aff}}\hat{\boxtimes} R/R}$$

and since

$$\Omega^{\mathrm{cont},\cdot}_{R^{\mathrm{aff}}\hat{\boxtimes} R/R} \cong \Omega^{\cdot}_{R^{\mathrm{aff}}} \hat{\boxtimes} R$$

we obtain a quasi-isomorphism

$$R \to \Omega^{\cdot}_{R^{\mathrm{aff}}} \hat{\boxtimes} R$$

Tensoring on the right by the finitely generated projective R-modules $F^p T^{\mathrm{poly},\cdot}(R)$ and $F^p D^{\mathrm{poly},\cdot}(R)$ gives what we want. □

9.4. **Tying it all together.**

We have a commutative diagram of DG-Lie algebras and (vertical) L_∞-maps

(9.13)

$$\begin{array}{ccccccc}
T^{\mathrm{poly},\cdot}(R) & \longrightarrow & \Omega_{R^{\mathrm{aff}}} \hat{\boxtimes} T^{\mathrm{poly},\cdot}(R) & \longrightarrow & \Omega_{R^{\mathrm{coord}}} \hat{\boxtimes} T^{\mathrm{poly},\cdot}(R) & \xrightarrow{\cong} & \Omega_{R^{\mathrm{coord}}} \hat{\otimes} T^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]]) \\
 & & \Big\downarrow \mathcal{V}^{\mathfrak{s}} & & \Big\downarrow \mathcal{V} & & \Big\downarrow \bar{\mathcal{U}}_{\omega_{MC}} \\
D^{\mathrm{poly},\cdot}(R) & \longrightarrow & \Omega_{R^{\mathrm{aff}}} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R) & \longrightarrow & \Omega_{R^{\mathrm{coord}}} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R) & \xrightarrow[\cong]{} & \Omega_{R^{\mathrm{coord}}} \hat{\otimes} D^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]])
\end{array}$$

where $\mathcal{V}$ is obtained from $\overline{\mathcal{U}}_{\omega_{MC}}$ using the horizontal isomorphisms and $\mathcal{V}^{\mathfrak{s}}$ is obtained from $(\overline{\mathcal{U}}_{\omega_{MC}})^{\mathfrak{s}}$ (see Lemmas 9.2.1,9.2.2). By Theorem 9.3.1 we know that the left most horizontal maps are quasi-isomorphisms.

Theorem 9.4.1. *The induced map*

$$\mu : T^{poly,\cdot}(R) \to H^{\cdot}(D^{poly,\cdot}(R))$$

is an isomorphism. If R has a system of parameters $(x_i)_i$ then

$$\mu(\partial_{i_1} \wedge \cdots \wedge \partial_{i_n}) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \partial_{i_{\sigma(1)}} \otimes \cdots \otimes \partial_{i_{\sigma(n)}} \tag{9.14}$$

where $\partial_i = \partial/\partial x_i$.

Proof. Since everything is local on R we may assume that R has a system of parameters. Denote the map defined by (9.14) by μ'. Since μ' is a quasi-isomorphism [36] it is sufficient to prove that $\mu = \mu'$.

Let us regard the complexes occurring in the (9.13) as double complexes such that the rows are obtained from the De Rham complexes. Assume $\gamma \in (\Omega_{R^{\mathrm{aff}}} \hat{\otimes} T^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]]))_{pq}$ where p is the column index.

According to (7.8) $\bar{\mathcal{U}}_{\omega_{MC},1}$ is given by

$$\bar{\mathcal{U}}_{\omega_{MC},1}(\gamma) = \sum_{j\geq 0} \frac{1}{j!} \bar{\mathcal{U}}_{j+1}(\omega^j_{MC}\gamma)$$

and one checks

$$\bar{\mathcal{U}}_{j+1}(\omega^j_{MC}\gamma) \in (\Omega_{R^{\mathrm{coord}}} \hat{\otimes} D^{\mathrm{poly},\cdot}(k[[t_1,\ldots,t_d]]))_{p+j,q-j} \tag{9.15}$$

(see for example the proof of Lemma 9.1.2). Since the horizontal maps in (9.13) are inclusions we obtain that $\mathcal{V}_1^{\mathfrak{s}}$ maps $(\Omega^\cdot_{R^{\mathrm{aff}}}\hat{\boxtimes}T^{\mathrm{poly},\cdot}(R))_{pq}$ to $\oplus_j(\Omega^\cdot_{R^{\mathrm{aff}}}\hat{\boxtimes}D^{\mathrm{poly},\cdot}(R))_{p+j,q-j}$.

We claim that the component corresponding to $j=0$ of $\mathcal{V}_1^{\mathfrak{s}}$ is equal to (the linear extension of) μ'. To prove this we look at the component of $\bar{\mathcal{U}}_{\omega_{MC},1}$ corresponding to $j=0$, which is equal to $\bar{\mathcal{U}}_1$. As discussed in §9.1 the vertical arrows in (9.13) are linear for the action of the De Rham complexes. Hence it suffices to prove that the following diagram is commutative

$$\begin{array}{ccc} T^{\mathrm{poly},n}(R) & \xrightarrow{\;i\;} & R^{\mathrm{coord}} \hat{\otimes} T^{\mathrm{poly},n}(k[[t_1,\ldots,t_d]]) \\ {\scriptstyle \mu'}\downarrow & & \downarrow{\scriptstyle \bar{\mathcal{U}}_1} \\ D^{\mathrm{poly},n}(R) & \xrightarrow[j]{} & R^{\mathrm{coord}} \hat{\otimes} D^{\mathrm{poly},n}(k[[t_1,\ldots,t_d]]) \end{array} \tag{9.16}$$

For convenience we have denoted the horizontal arrows by i and j. They are obtained from the "expansion in local coordinates" isomorphism (6.12)

$$R^{\mathrm{coord}} \hat{\boxtimes} R \xrightarrow{\cong} R^{\mathrm{coord}} \hat{\otimes} k[[t_1,\ldots,t_d]]$$

Thus a poly-differential operator or vector field on R is linearly extended to one on $R^{\mathrm{coord}} \hat{\boxtimes} R$ and then transported to an operator on $R^{\mathrm{coord}} \hat{\otimes} k[[t_1,\ldots,t_d]]$. Clearly i and j are compatible with cup-product.

To avoid confusion we write ∂_{x_i} for $\partial/\partial x_i$ and ∂_{t_j} for $\partial/\partial t_j$. According to for example the proof of Theorem 6.1.4 the matrix $(\partial x_i/\partial t_j)_{ij}$ is an invertible matrix over $R^{\mathrm{coord}}[[t_1,\ldots,t_d]]$. Denote the inverse matrix by $(\partial t_j/\partial x_i)_{ij}$. Then

$$i(\partial_{x_{i_1}} \wedge \cdots \wedge \partial_{x_{i_n}}) = \sum_{j_1,\ldots,j_n} \frac{\partial t_{j_1}}{\partial x_{i_1}} \cdots \frac{\partial t_{j_n}}{\partial x_{i_n}} \partial_{t_{j_1}} \wedge \cdots \wedge \partial_{t_{j_n}}$$

$$j(\partial_{x_{i_1}} \otimes \cdots \otimes \partial_{x_{i_n}}) = \sum_{j_1,\ldots,j_n} \frac{\partial t_{j_1}}{\partial x_{i_1}} \cdots \frac{\partial t_{j_n}}{\partial x_{i_n}} \partial_{t_{j_1}} \otimes \cdots \otimes \partial_{t_{j_n}}$$

Comparing the formulas (8.4) and (9.14) (which we have taken to define μ') we see that (9.16) is indeed commutative.

The first component of an L_∞-map always commutes with the differential thus we have a map of complexes

$$\mathcal{V}_1^{\mathfrak{s}} : \Omega^\cdot_{R^{\mathrm{aff}}} \hat{\boxtimes} T^{\mathrm{poly},\cdot}(R) \to \Omega^\cdot_{R^{\mathrm{aff}}} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R)$$

We filter the two complexes according to the column index. By (9.15) this filtration is compatible with $\mathcal{V}_1^{\mathfrak{s}}$ and the graded map associated to $\mathcal{V}_1^{\mathfrak{s}}$ is $(\mathcal{V}_1^{\mathfrak{s}})_{j=0}$, which we have shown to be equal to the linear extension of μ'.

Denote by H^{columns} the homology of the columns of a double complex. We clearly have

$$H^{\mathrm{columns}}(\Omega^\cdot_{R^{\mathrm{aff}}} \hat{\boxtimes} T^{\mathrm{poly},\cdot}(R)) = \Omega^\cdot_{R^{\mathrm{aff}}} \hat{\boxtimes} T^{\mathrm{poly},\cdot}(R)$$

and since $D^{\mathrm{poly},\cdot}(R)$ consists of filtered projective R-modules with filtered projective homology we also have

$$H^{\mathrm{columns}}(\Omega^\cdot_{R^{\mathrm{aff}}} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R)) = \Omega^\cdot_{R^{\mathrm{aff}}} \hat{\boxtimes} H^\cdot(D^{\mathrm{poly},\cdot}(R))$$

Taking homology for the rows (and using Theorem 9.3.1) we see that $\mathcal{V}^{\mathfrak{s}}$ induces indeed μ' on homology. □

9.5. The global case.

Theorem 9.5.1. *There exists a sheaf of DG-Lie algebras* $\mathfrak{l}[1]$ *on* X *together with* L_∞ *morphisms*

$$\mathcal{T}_X^{poly,\cdot}[1] \to \mathfrak{l}[1] \leftarrow \mathcal{D}_X^{poly,\cdot}[1]$$

Furthermore

1. $\mathfrak{l}$ *as well as the given quasi-isomorphisms do not depend on any choices.*[3]
2. *If* X *has a system of parameters* $(x_i)_i$ *then the resulting map on homology*

$$\mathcal{T}_X^{poly,\cdot} \to H^\cdot(\mathcal{D}_X^{poly,\cdot})$$

is given by the HKR-formula.

$$\partial_{i_1} \wedge \cdots \wedge \partial_{i_n} \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \partial_{i_{\sigma(1)}} \otimes \cdots \otimes \partial_{i_{\sigma(n)}}$$

where $\partial_i = \partial_{x_i}$.

Proof. Since all our constructions are canonical we may assume that $X = \operatorname{Spec} R$ where R is smooth of dimension d.

The diagram (9.13) in combination with Theorem 9.4.1 furnishes us with L_∞-quasi-isomorphisms

$$T^{\mathrm{poly},\cdot}(R) \to \Omega_{R^{\mathrm{aff}}} \hat{\boxtimes} T^{\mathrm{poly},\cdot}(R) \xrightarrow{\mathcal{V}^s} \Omega_{R^{\mathrm{aff}}} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R) \leftarrow D^{\mathrm{poly},\cdot}(R)$$

We now take $\mathfrak{l}$ to be equal to $\Omega_{R^{\mathrm{aff}}} \hat{\boxtimes} D^{\mathrm{poly},\cdot}(R)$. (2) follows directly from Theorem 9.4.1. □

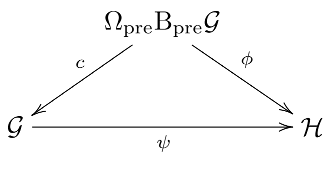
Proof of Theorem 1.1. Given Theorem 9.5.1 we only need to prove that if we have a L_∞ quasi-isomorphism

$$\psi : \mathcal{G} \to \mathcal{H}$$

between sheaves of DG-Lie algebras then $\mathcal{G}$ and $\mathcal{H}$ are isomorphic in the homotopy category of DG-Lie algebras. This is done in the standard way using the bar-cobar construction [20, 24, 31].

The bar-cobar construction may be performed in any symmetric abelian monoidal category, in particular it can be done in the categories of presheaves and sheaves of vector spaces. Since the bar-cobar construction involves only colimits it is compatible with sheaffification.

Considering ψ first as a morphism of presheaves there is a commutative diagram of L_∞-morphisms of presheaves of DG-Lie algebras



where c, ϕ' are morphism of presheaves of DG-Lie algebras and c is a quasi-isomorphism.

[3]Except for the choice of the L_∞-quasi-isomorphism in the formal case satisfying the properties (P4)(P5)

Sheaffifying we obtain an analogous diagram of sheaves of DG-Lie algebras

$$\begin{array}{ccc} & \Omega\mathrm{B}\mathcal{G} & \\ {\scriptstyle\tilde{c}}\swarrow & & \searrow{\scriptstyle\tilde{\phi}} \\ \mathcal{G} & \xrightarrow[\psi]{} & \mathcal{H} \end{array}$$

Since sheaffication is exact $\tilde{c}$ is still a quasi-isomorphism. Since ψ is a quasi-isomorphism by assumption we obtain that $\tilde{\phi}$ is a quasi-isomorphism as well. We conclude that $\mathcal{G}$ and $\mathcal{H}$ are isomorphic in the homotopy category of sheaves of DG-Lie algebras. □

Appendix A. Reminder on the Thom-Sullivan normalization

The material in this section is standard. See e.g. [19]. Let k be a field. Let S be a small category and let $M, N : S \to \mathrm{Mod}(k)$ be respectively a contravariant and a covariant functor. We define $M\underline{\otimes}_S N$ as the subset of $\prod_{P\in \mathrm{Ob}(S)} M(P)\otimes_k N(P)$ consisting of $(c_P)_P$ such that for all $\phi : P \to Q$ we have

$$(M(\phi)\otimes 1)(c_Q) = (1\otimes N(\phi))(c_P)$$

inside $M(P)\otimes_k N(Q)$. We extend the bifunctor $-\underline{\otimes}-$ in the obvious way to the case where M, N take values in complexes.

We will now consider the case where is S is the simplicial category Δ. If $N(\Delta[n])$ denotes the normalized (combinatorial) cochain complex of $\Delta[n]$ then

$$N(-) : \Delta[n] \mapsto N(\Delta[n])$$

is a contravariant functor from Δ to $C(k)$.

If now A is a cosimplicial k-vector space then we may consider

$$N(-)\underline{\otimes}_\Delta A$$

The following is well-known

Proposition A.1. *$N(-)\underline{\otimes}_\Delta A$ is canonically isomorphic to the normalized cochain complex $N(A)$ (given by the common kernels of the degeneracies) of A.*

Now fix a k-linear DG-operad $\mathcal{O}$. If A is a cosimplicial $\mathcal{O}$-algebra then $N(A)$ will in general not have the structure of an $\mathcal{O}$-algebra. The Thom-Sullivan construction repairs this defect. The idea is to replace the complexes $N(\Delta[n])$ in Proposition A.1 by quasi-isomorphic complexes which have the structure of a commutative DG-algebra.

We now assume that k has characteristic zero. Think of $\Delta[n]$ as the affine space

$$\operatorname{Spec} k[t_0, \dots, t_n]/(t_0 + \cdots + t_n - 1)$$

Taking the algebraic De Rham complex of $\Delta[n]$ yields a contravariant functor $\Omega^{\cdot}(-)$ from Δ to commutative DG-algebras. The Thom-Sullivan normalization of a cosimplicial $\mathcal{O}$-algebra $A^{\cdot}$ is defined as

$$N(A)^{TS} = \Omega^{\cdot}(-)\underline{\otimes}_\Delta A$$

From the commutativity of $\Omega^{\cdot}(\Delta[n])$ it easily follows that $N(A)^{TS}$ has a canonical structure as an $\mathcal{O}$-algebra.

Proposition A.2. [8] *There is a canonical quasi-isomorphism (as complexes of k-vector spaces)*

$$N(A)^{TS} \to N(A)$$

The quasi-isomorphism is constructed using functorial homotopy equivalences $\Omega^\cdot(\Delta[n]) \to N(\Delta[n])$. The latter are obtained by integrating differential forms.

If A is a complex of k-vector spaces then we may consider A as a constant cosimplicial object. One easily checks that

$$A \cong N(A)^{\mathrm{TS}} \cong N(A) \tag{A.1}$$

Appendix B. Derived global sections of sheaves of algebras

B.1. **Introduction.** Let X be a topological space and fix a k-linear DG-operad $\mathcal{O}$ for a field of characteristic zero. If $\mathcal{A}$ is a sheaf of $\mathcal{O}$-algebras on X then it is easy to see that $H^\cdot(X,\mathcal{A})$ has the structure of an $H^\cdot(\mathcal{O})$-algebra. However one would like to give $R\Gamma(X,\mathcal{A})$ the structure of an $\mathcal{O}$-algebra as well.

In [21] Hinich constructs a model structure on the *presheaves* of $\mathcal{O}$-algebras on X which is such that a presheaf of $\mathcal{O}$-algebras is weakly equivalent to its sheaffification.

It follows from Hinich's construction that $R\Gamma(X,\mathcal{A})$ is quasi-isomorphic to $\Gamma(X,\mathcal{A}')$ for an arbitrary fibrant resolution $\mathcal{A} \to \mathcal{A}'$. In this way we obtain indeed an actual $\mathcal{O}$-algebra representing $R\Gamma(X,\mathcal{A})$.

Note however that the choice of $\mathcal{A}'$ is not functorial[4] and furthermore it depend on the operad $\mathcal{O}$. In this appendix we give an alternative construction for the algebra structure on $R\Gamma(X,\mathcal{A})$ (if $\mathcal{A}$ has left bounded cohomology) which is functorial and whose outcome does not depend on $\mathcal{O}$. More precisely: for a complex of sheaves $\mathcal{A}$ on X we construct a complex $R\Gamma(X,\mathcal{A})^{\mathrm{tot}}$ which is functorial in $\mathcal{A}$ and which inherits any algebra structure present on $\mathcal{A}$.

Our construction is a generalization of a construction originally due to Hinich and Schechtman which first replaces $\mathcal{A}$ by a (DG-)cosimplicial algebra [22, 23] using the Čech construction. The Thom-Sullivan normalization (see Appendix A) is then used to transform this cosimplicial algebra into a genuine algebra over $\mathcal{O}$. It is clear that this procedure has the properties mentioned in the previous paragraph.

The Hinich-Schechtman construction works well for quasi-coherent sheaves but must be modified in more general situations. This issue is not entirely academic as non-quasi-coherent sheaves do occur in nature. Examples in this paper are $\mathfrak{l}$ (Theorem 9.5.1) and $\Omega\mathrm{B}\mathcal{G}$ (the proof of Theorem 1.1).

Our initial idea was to replace Čech cohomology by a colimit over all hypercoverings of X but as the category of hypercoverings is only filtered in a homotopy theoretic sense [2], this creates rather unpleasant technical difficulties. Luckily it seems we can make at least some of these difficulties go away by replacing hypercoverings with pro-hypercoverings, which is what we will do in this section.

Although below we will work in an arbitrary Grothendieck topos, for simplicity we will, in this introduction, continue to use the topological space X. Let $\mathrm{Alg}^+(X,\mathcal{O})$ the category of $\mathcal{O}$-algebra objects in $\mathrm{Sh}(X)$ with left bounded cohomology and let $\mathrm{Alg}(\mathcal{O})$ be the category of $\mathcal{O}$-algebras. We equip both categories with weak equivalences given by quasi-isomorphisms. We will construct a functor (see §B.8)

$$\Sigma : \mathrm{Alg}^+(X,\mathcal{O}) \to \Delta\,\mathrm{Alg}(\mathcal{O})$$

[4] It is of course functorial in a homotopy theoretic sense

such that the cochain complex associated to $\Sigma(\mathcal{A})$ for $\mathcal{A} \in \mathrm{Alg}^+(X, \mathcal{O})$ is canonically isomorphic to the derived global sections of $\mathcal{A}$ when viewed as a complex of sheaves of abelian groups. This part does not require the presence of a basefield of characteristic zero.

For the benefit of the reader we indicate how Σ is defined. We will construct a pro-object $F = (F_\alpha)_\alpha$ in the category of hypercovering of X which is *homotopy projective* (a suitable lifting property, see (B.12)) and we put

$$\Sigma(\mathcal{A}) = \operatorname{inj\,lim}_{\alpha} \operatorname{Hom}(F_\alpha, \mathcal{A}) \tag{B.1}$$

We show in Proposition B.6.3 that F is unique up to unique isomorphism in a homotopy theoretic sense. This implies that Σ is defined up to a unique natural isomorphism when viewed as a functor between homotopy categories.

It follows from Proposition A.2 that if $\mathcal{O}$ is k-linear for k a field of characteristic zero and we put

$$R\Gamma(X, -)^{\mathrm{tot}} = N(\Sigma(-))^{\mathrm{TS}}$$

then we obtain a functor

$$R\Gamma(X, -)^{\mathrm{tot}} : \mathrm{Alg}^+(X, \mathcal{O}) \to \mathrm{Alg}(\mathcal{O})$$

such that the underlying complex of vector spaces of $R\Gamma(X, \mathcal{A})^{\mathrm{tot}}$ is isomorphic to $R\Gamma(X, \mathcal{A})$ in $D(k)$.

In the last section of this appendix we outline the connection of our construction with that of Hinich in [21].

B.2. **Simplicial objects.** In this section we recall some standard constructions on simplicial objects. Let $\mathcal{P}$ be a category with arbitrary limits and colimits. We consider the category $\Delta^\circ\mathcal{P}$ of simplicial objects in $\mathcal{P}$. If $F \in \mathcal{P}$ then we denote by $\hat{F}$ the associated constant simplicial object. $F \mapsto \hat{F}$ is a left adjoint to the functor $F \mapsto F_0$.

We may define a bifunctor

$$- \times - : \Delta^\circ\mathcal{P} \times \Delta^\circ \mathrm{Set} \to \Delta^\circ\mathcal{P} : (F, S) \mapsto (F_n \times S_n)_n \tag{B.2}$$

where $F_n \times S_n$ is the $|S_n|$-fold coproduct of F_n. If $F \in \mathcal{P}$ then we define $F \times S$ as $\hat{F} \times S$. It is easy to see that any object $F \in \Delta^\circ\mathcal{P}$ is a coequalizer of the form

$$\textstyle\coprod_{[i]\to[j]\in\Delta} F_j \times \Delta[i] \rightrightarrows \coprod_i F_i \times \Delta[i] \longrightarrow F \tag{B.3}$$

The functor $\mathcal{P} \times \Delta^\circ \mathrm{Set} \to \Delta^\circ\mathcal{P} : (F, S) \mapsto \hat{F} \times S$ has a right adjoint in its second argument given by a bifunctor.

$$(\Delta^\circ \mathrm{Set})^\circ \times \Delta^\circ\mathcal{P} \to \mathcal{P} : (S, F) \mapsto \operatorname{Hom}(S, F) \tag{B.4}$$

which is the unique functor such that $\operatorname{Hom}(-, F)$ sends colimits to limits and $\operatorname{Hom}(\Delta[n], F) = F_n$.

The associated derived functor

$$(\Delta^\circ \mathrm{Set})^\circ \times \Delta^\circ\mathcal{P} \to \Delta^\circ\mathcal{P} : (S, F) \mapsto \underline{\operatorname{Hom}}(S, F)$$

defined by

$$\underline{\operatorname{Hom}}(S, F)_n = \operatorname{Hom}(\Delta[n] \times S, F)$$

is the right adjoint in the second argument to (B.2).

For $F \in \Delta^\circ\mathcal{P}$ write $F \times I = F \times \Delta[1]$ (the *cylinder* object of F) and $F^I = \underline{\mathrm{Hom}}(\Delta[1], F)$ (the *path* object of F).

The category $\Delta^\circ\mathcal{P}$ is enriched in simplicial sets (it is a so-called simplicial category). Let $F, G \in \Delta^\circ\mathcal{P}$. Then the simplicial set $\underline{\mathrm{Hom}}_{\Delta^\circ\mathcal{P}}(F, G)$ is defined by

$$\underline{\mathrm{Hom}}_{\Delta^\circ\mathcal{P}}(F, G)_n = \mathrm{Hom}_{\Delta^\circ\mathcal{P}}(F \times \Delta[n], G)$$

Define the homotopy category $\mathrm{Ho}(\Delta^\circ\mathcal{P})$ of $\Delta^\circ\mathcal{P}$ by

$$\mathrm{Hom}_{\mathrm{Ho}(\Delta^\circ\mathcal{P})}(F, G) = \text{connected components of } \underline{\mathrm{Hom}}_{\Delta^\circ\mathcal{P}}(F, G) \tag{B.5}$$

In the sequel we will use the terminology exhibited in the next definition.

Definition B.2.1. (1) Two maps $f_{0,1} : F \to G$ in $\Delta^\circ\mathcal{P}$ are *strictly homotopic* if there is a map $f' : F \times I \to G$ such that the f_i is the composition of $F = F \times \Delta[0] \xrightarrow{\partial^i} F \times \Delta[1] = F \times I \xrightarrow{f'} G$.

(2) Two maps $f_{0,1} : F \to G$ in $\Delta^\circ\mathcal{P}$ are *combinatorially homotopic* if they can be connected by a chain of strict homotopies and their inverses, or equivalently if they represent the same maps in $\mathrm{Ho}(\Delta^\circ\mathcal{P})$.

(3) A map $f : F \to G$ in $\Delta^\circ\mathcal{P}$ is a *combinatorial homotopy equivalence* if there is a map $g : G \to F$ such that fg and gf are combinatorially homotopy equivalent to the identity, or equivalently if f is invertible in $\mathrm{Ho}(\Delta^\circ\mathcal{P})$.

Lemma B.2.2. *Let $F \in \Delta^\circ\mathcal{P}$. Then the functors*

$$F \times - : \Delta^\circ\,\mathrm{Set} \to \Delta^\circ\mathcal{P}$$

$$\underline{\mathrm{Hom}}(-, F) : \Delta^\circ\,\mathrm{Set} \to \Delta^\circ\mathcal{P}$$

preserve strict homotopy equivalent maps (and hence also combinatorially homotopic maps and combinatorial homotopy equivalences).

Proof. Let us consider the second functor. Let $f' : S \times I \to T$ be a homotopy between maps $f_0, f_1 : S \to T$ between simplicial sets. Let $\tilde{f}'$, $\tilde{f}_0, \tilde{f}_1$ be the maps obtained applying $\underline{\mathrm{Hom}}(-, F)$. Since

$$\underline{\mathrm{Hom}}(S \times I, F) = \underline{\mathrm{Hom}}(S, F)^I$$

we obtain that $\tilde{f}'$ is a map $\underline{\mathrm{Hom}}(T, F) \to \underline{\mathrm{Hom}}(S, F)^I$ which yields a map $\underline{\mathrm{Hom}}(T, F) \times I \to \underline{\mathrm{Hom}}(S, F)$. It is easy to see that this is a homotopy between $\tilde{f}_0$ and $\tilde{f}_1$. □

Corollary B.2.3. *The "constant path" map $F \to F^I$ is a combinatorial homotopy equivalence.*

Proof. This follows from the fact that it is obtained from the combinatorial homotopy equivalence $\Delta[1] \to \Delta[0]$ in $\Delta^\circ\,\mathrm{Set}$. □

The following is standard.

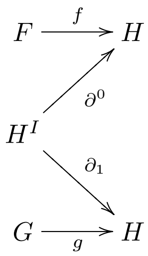
Lemma B.2.4. *Assume that $\mathcal{Q}$ is abelian. For $F \in \Delta^\circ\mathcal{Q}$ let $C_*(F)$ be the usual (unnormalized) chain complex of F. If $f, g : F \to G$ are strictly homotopic maps in $\Delta^\circ\mathcal{P}$ then $C_*(f)$ and $C_*(g)$ are homotopic.*

The following is standard as well.

Lemma B.2.5. *Let $f_0, f_1 : F \to G$ be strictly homotopic maps in $\Delta^\circ\mathcal{P}$. Then the induced maps $\mathbb{Z}f_0, \mathbb{Z}f_1 : \mathbb{Z}F \to \mathbb{Z}G$ are strictly homotopic as well.*

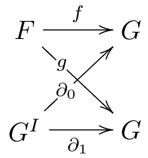
Proof. f_0, f_1 are induced from a homotopy $f' : F \times I \to G$. Since the functor $W \mapsto \mathbb{Z}W$ is a left adjoint it commutes with coproduct. Hence $\mathbb{Z}(F \times I) = (\mathbb{Z}F) \times I$. Thus f' yields a homotopy in $\Delta^\circ\mathcal{P}_\mathbb{Z}$, $\mathbb{Z}f' : (\mathbb{Z}F) \times I \to \mathbb{Z}G$. It is easy to see that $\mathbb{Z}f'$ induces $\mathbb{Z}f_0$, $\mathbb{Z}f_1$. □

Definition B.2.6. Let $f : F \to H$, $g : G \to H$ be in $\Delta^\circ\mathcal{P}$. The *homotopy fiber product* $F \overset{h}{\times}_H G$ is the limit of the following diagram.



If $p_0 : F \overset{h}{\times}_H G \to F$, $p_1 : F \overset{h}{\times}_H G \to G$ are the resulting projection maps then clearly $f \circ p_0$ and $g \circ p_1$ are strictly homotopic.

Definition B.2.7. Similarly if $f, g : F \to G$ are maps in $\Delta^\circ\mathcal{P}$ then we define the *homotopy equalizer* of f and g as the limit of the following diagram



Let $\Delta^{\leq n}$ be the simplicial category truncated in dimension n and let $(-)_{\leq n}$ denote the truncation functor $\Delta^\circ\mathcal{P} \to \Delta^{\leq n,\circ}\mathcal{P}$. The right adjoint to $(-)_{\leq n}$ is the coskeleton functor denoted by cosk_n. Concretely

$$(\operatorname{cosk}_n G)_m = \operatorname{Hom}(\Delta[m]_{\leq n}, G)$$

The truncation functor also has a left adjoint which is denoted by sk_n. If $F \in \Delta^{\leq n,\circ}\mathcal{P}$ then using the truncated version of (B.3) we see that $\operatorname{sk}_n F$ is the coequalizer in $\Delta^\circ\mathcal{P}$ of

$$\coprod_{[i]\to[j]\in\Delta^{\leq n}} F_j \times \Delta[i] \rightrightarrows \coprod_{i\leq n} F_i \times \Delta[i]$$

As is customary we will also use the the notations sk_n, cosk_n for the compositions $\operatorname{sk}_n \circ (-)_{\leq n}$, $\operatorname{cosk}_n \circ (-)_{\leq n}$.

B.3. **Grothendieck topoi.** From here on $\mathcal{P}$ is a Grothendieck topos [2]. This means that $\mathcal{P}$ has properties very reminiscent of those of the category of sets. By Giraud's theorem [2] $\mathcal{P}$ may be realized as the category of sheaves on a small site $\mathcal{C}$. Therefore we sometimes we refer to the objects of $\mathcal{P}$ as "sheaves". Recall that a *site* is a category $\mathcal{C}$ equipped with a so-called *Grothendieck topology*. I.e. for every $A \in \mathcal{C}$ there is a collection $\mathcal{T}_A$ of subfunctors of $\mathcal{C}(-, A)$ (called coverings) satisfying the axioms of [2, Def I.1.1].

We recall the following standard result (see e.g. [28, Prop. 6.20]).

Lemma B.3.1. *Let $\mathcal{C}$ be a small site and let* $\mathrm{Pre}(\mathcal{C})$ *and* $\mathrm{Sh}(\mathcal{C})$ *be respectively the categories of presheaves and sheaves on $\mathcal{C}$. Let $a : \mathrm{Pre}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ be the sheaffification functor. For $F \in \mathrm{Pre}(\mathcal{C})$ define $|F| = \sum_{C\in\mathcal{C}} |F(C)|$. Let $|\mathcal{C}|$ be the sum of the cardinalities of the* Hom*'sets in $\mathcal{C}$. Then we have the following bound*

$$|aF| \le |\mathcal{C}|(2|F|)^{|\mathcal{C}|} \tag{B.6}$$

If $\mathfrak{a} = 2^{\mathfrak{b}}$ where $\mathfrak{b} \ge \max(|\mathcal{C}|, |\mathbb{N}|)$ then $|F| \le \mathfrak{a}$ implies $|aF| \le \mathfrak{a}$.

Proof. To prove (B.6) we may assume that F is separated. Indeed if we identify sections in F which are locally identical then we only reduce $|F|$.

So assume that F is separated. For any $P \in \mathcal{C}$ we have

$$(aF)(P) = \operatorname{inj\,lim} \operatorname{Hom}_{R\in\mathcal{T}_P}(R, F)$$

Thus

$$|aF| \le \sum_{P\in\mathcal{C}} |\operatorname{Hom}_{R\in\mathcal{T}_P}(R, F)|$$

Since the existence of identities implies $|\operatorname{Ob}(\mathcal{C})| \le |\mathcal{C}|$ we deduce from this

$$|aF| \le |\mathcal{C}| 2^{|\mathcal{C}|} |F|^{|\mathcal{C}|}$$

which yields (B.6).

Thus if $|F| \le \mathfrak{a}$ with $\mathfrak{a}$ as in the statement of the lemma.

$$|aF| \le \mathfrak{a}^{|\mathcal{C}|} = 2^{\mathfrak{b}|\mathcal{C}|} = 2^{\mathfrak{b}} = \mathfrak{a} \quad \square \tag{B.7}$$

Lemma B.3.2. *Let $\mathcal{P}$ be the category of sheaves over a small site $\mathcal{C}$. Put*

$$\mathcal{P}_{\mathfrak{a}} = \{F \in \mathcal{P} \mid |F| \le \mathfrak{a}\}$$

where $\mathfrak{a}$ is as in Lemma B.3.1. Then $\mathcal{P}_{\mathfrak{a}}$ is closed under finite limits, finite colimits, epimorphisms and monomorphisms.

Furthermore $\mathcal{P}_{\mathfrak{a}}$ satisfies the following cofinality property. For any epimorphism $f : F \to G_0$ with $G_0 \in \mathcal{P}_{\mathfrak{a}}$ there exists a map $F_0 \to F$ such that $F_0 \in \mathcal{P}_{\mathfrak{a}}$ and the composed map $F_0 \to G_0$ is an epimorphism

Proof. This follows easily from Lemma B.3.1 and the corresponding results for presheaves. □

Let $\mathcal{P}_{\mathbb{Z}}$ be the category of abelian group objects in $\mathcal{P}$.

Lemma B.3.3. (1) *$\mathcal{P}_{\mathbb{Z}}$ is a Grothendieck category.*
(2) *The forgetful functor $\mathcal{P}_{\mathbb{Z}} \to \mathcal{P}$ has a left adjoint.*
(3) *If $F, G \in \mathcal{P}_{\mathbb{Z}}$ then the functor of bilinear maps* $\mathrm{Bilin}(F \times G, -)$ *is representable by an object $F \otimes G$. In this way $\mathcal{P}_{\mathbb{Z}}$ becomes a symmetric monoidal category.*

Proof. These facts may be proved by realizing $\mathcal{P}$ as the category of sheaves on a small site $\mathcal{C}$. Then $\mathcal{P}_{\mathbb{Z}}$ is precisely the category of sheaves of abelian groups and the statements are standard. □

We will denote the left adjoint to $\mathcal{C}_{\mathbb{Z}} \to \mathcal{C}$ by $F \mapsto \mathbb{Z}F$. If e is the final object of $\mathcal{P}$ then we write $\underline{\mathbb{Z}}$ for $\mathbb{Z}e$.

B.4. **Hypercoverings.** We recall briefly some results about hypercoverings. An object F in $\Delta^\circ\mathcal{P}$ is a *hypercovering* if for all m the the canonical morphism

$$F_{m+1} \to (\operatorname{cosk}_m F)_{m+1} \tag{B.8}$$

is an epimorphism (see e.g. [14, §1.1]) and if the map of F_0 to the final object e or $\mathcal{P}$ is an epimorphism as well. Similarly $F \in \Delta^{\le n \circ}\mathcal{P}$ is a truncated hypercovering if (B.8) holds for $m \le n-1$.

There are many equivalent characterizations for the notion of a hypercovering. Put $\partial\Delta[n] = \operatorname{sk}_{n-1}\Delta[n]$. The following is a direct translation of the definition. F is a hypercovering if and only if for all n the morphism

$$\operatorname{Hom}(\Delta[n], F) \to \operatorname{Hom}(\partial\Delta[n], F) \tag{B.9}$$

is an epimorphism and if $F_0 \to e$ is an epimorphism. This is called the *local lifting property*. From this way of writing the definition we see that if $\mathcal{P}$ has enough points [2] then F is a hypercovering if and only for every point p the simplicial set $(p^* F_n)_n$ is non-empty, acyclic and Kan.

Remark B.4.1. The definition of hypercovering we use is in fact a slight modification of the one used by Verdier (which depends on a site representing $\mathcal{P}$). For the original definition see §B.10 below.

The following result follows from [2, Lemma V.7.2.1].

Proposition B.4.2. *Let $F \in \Delta^\circ\mathcal{P}$ be a hypercovering. Then the chain complex $C_*(\mathbb{Z}F)$ associated to $\mathbb{Z}F$ is acyclic in degrees > 0 and its cohomology is equal to $\underline{\mathbb{Z}}$ in degree zero.*

Note that this result is clear if $\mathcal{P}$ has enough points since in that case we may check it on stalks (see [3]).

We will frequently use the following results which are proved in the same way as for acyclic Kan simplicial sets. In case $\mathcal{P}$ has enough points, they can also be checked on stalks.

Proposition B.4.3.
1. *Let F be a hypercovering and S a finite simplicial set (i.e. S has only a finite number of non-degenerate simplices). Then $\underline{\operatorname{Hom}}(S, F)$ is a hypercovering. In particular the path object of F is a hypercovering.*
2. *Homotopy fiber products and homotopy equalizers of hypercoverings are hypercoverings.*

We quote some results from [3].

Proposition B.4.4. [3] *Let G be a hypercovering and let $\psi_0 : F_0 \to G_{\le n}$ be a morphism of hypercoverings trunctated in degree n. Then there is a hypercovering F and a morphism of hypercoverings $\psi : F \to G$ such that $\psi_{\le n}$ is equal to ψ_0.*

If $F \in \Delta^\circ\mathcal{P}$ then $D_n(F) = \cup_{\sigma:[n]\to[m] \text{ surj}, m<n} \sigma F_m \subset F_n$. We call $D_n(F)$ the *degenerate part* of F_n. We say that F is *split* in degree n if $D_n(F)$ has a (necessarily unique) complement $N_n(F)$ in $\mathcal{P}$. $N_n(F)$ (if existing) is the *non-degenerate* part of F_n.

If F is split up to degree n then one may write

$$F_n = \coprod_{\sigma:[n]\to[m] \text{ surj}} \sigma N_m(F)$$

The following proposition shows that we may restrict ourselves to split hypercoverings, if necessary.

Proposition B.4.5. [3] *Assume that G is a hypercovering in $\mathcal{P}$ split up to degree n. Then there exists a map $\psi : F \to G$ where F is a split hypercovering in $\mathcal{P}$ and $\psi_{\le n}$ is the identity.*

The next propositions shows that we may arbitrarily refine the non-degenerate part of a split hypercovering.

Proposition B.4.6. [3] *Let G be a split hypercovering in $\mathcal{P}$ and let $\phi : N' \to N_n(G)$ be an epimorphism. Then there exists a map $\psi : F \to G$ of split hypercoverings in $\mathcal{P}$ such that $\psi_{\le n-1}$ is the identity and furthermore $N_n(F) = N'$ in such a way that ψ_n restricts to the map ϕ.*

Throughout we fix a full small subcategory $\mathcal{P}_0$ of $\mathcal{P}$ which is closed under finite limits, finite colimits, monomorphisms and epimorphisms and which satisfies the cofinality condition of Lemma B.3.2. Such a $\mathcal{P}_0$ may be constructed by taking $\mathcal{P}_0$ to be a skeletal subcategory of some $\mathcal{P}_\mathfrak{a}$ where $\mathcal{P}_\mathfrak{a}$ is as in Lemma B.3.2.

$\mathcal{H}(\mathcal{P})$ is the category of hypercoverings in $\mathcal{P}$ and $\mathcal{H}(\mathcal{P}_0)$ is the full subcategory of hypercoverings F such that $F_n \in \mathcal{P}_0$ for all n.

Lemma B.4.7. *If $G \in \mathcal{H}(\mathcal{P})$ then there exists $F \in \mathcal{H}(\mathcal{P}_0)$ together with a morphism $F \to G$.*

Proof. We construct F step by step. Our first step is to select a map $F'_0 \to G$ such that the composition $F'_0 \to G_0 \to e$ is an epimorphism using Lemma B.3.2. We then extend F'_0 to a map of hypercoverings $F' \to G$ using Proposition B.4.4. Using Proposition B.4.5 we may assume that F' is split.

Assume now that we have constructed a map of hypercoverings $F' \to G$ such that $F'_i \in \mathcal{P}_0$ for $i \le n$. Assume in addition that F' is split.

Consider the epimorphism $F'_{n+1} \to (\operatorname{cosk}_n F')_{n+1}$. We have $(\operatorname{cosk}_n F')_{n+1} \in \mathcal{P}_0$ since the construction of the coskeleton involves only finite limits. Let N be the image of $N_{n+1}(F')$ in $\operatorname{cosk}_n F'$ and choose $N_0 \in \mathcal{P}_0$ together with a map $N_0 \to N_{n+1}(F')$ such that the composition $N_0 \to N_{n+1}(F') \to N$ is an epimorphism. Put

$$F''_{n+1} = N_0 \coprod \coprod_{\sigma:[n+1]\to[m] \text{ surj}, m\le n} \sigma N_m(F)$$

and extend the truncated hyperovering $F''_{n+1}, F_n, \dots, F_0$ to a hypercovering F'' mapping to F' using Proposition B.4.4. Then F'' coincides with F' in degrees $\le n$ and is in $\mathcal{P}_0$ in degrees $\le n+1$. Using Proposition B.4.5 we may assume that F'' is split. Repeating this procedure we ultimately construct the desired F. □

B.5. **Pro-objects.** Recall that if $\mathcal{D}$ is any category then $\operatorname{Pro}\mathcal{D}$ is the category with objects denoted by formal symbols $''\operatorname{proj\,lim}''_{\alpha\in I} A_i$ where I° is a (small) filtered category and A is a functor $I \to \mathcal{D}$. The Hom-sets are given by

$$\operatorname{Hom}_{\operatorname{Pro}\mathcal{D}}(''\operatorname{proj\,lim}''_{\alpha\in I} A_\alpha, ''\operatorname{proj\,lim}''_{\beta\in I} B_\beta) = \operatorname{proj\,lim}_\beta \operatorname{inj\,lim}_\alpha \operatorname{Hom}_\mathcal{D}(A_\alpha, B_\beta)$$

Below we will omit the quotes around proj lim.

By [3, §A.4.4] $\operatorname{Pro}\mathcal{D}$ is closed under filtered inverse limits. By [3, Cor. 3.3] any finite diagram $D \to \operatorname{Pro}\mathcal{D}$ where D is directed ("contains no loops") is the image of an object in $\operatorname{Pro}\operatorname{Fun}(D,\mathcal{D})$. Informally we say that the diagram can be constructed "levelwise". [3, Prop. A.4.1] states that limits and colimits of finite levelwise defined limits of pro-object can be computed levelwise as well.

Let $L(\mathcal{D})$ be the category of left exact covariant functors $\mathcal{D} \to \mathrm{Set}$. Then there is fully faithful embedding

$$\operatorname{Pro}\mathcal{D} \to L(\mathcal{D})^\circ : (A_\alpha)_\alpha \mapsto \operatorname*{inj\,lim}_\alpha \operatorname{Hom}_{\mathcal{D}}(A_\alpha, -) \tag{B.10}$$

The construction of filtered inverse limits in $\operatorname{Pro}\mathcal{D}$ in [3, Prop. A.4.4] shows that the functor (B.10) commutes with filtered inverse limits. In particular the objects in $\mathcal{D}$ are "cofinitely presented". Let $(F_i)_{i\in I}$ be a filtered inverse system of objects in $\operatorname{Pro}\mathcal{D}$ and $F \in \mathcal{D}$. Then

$$\operatorname{Hom}_{\operatorname{Pro}\mathcal{D}}(\operatorname*{proj\,lim}_i F_i, F) = \operatorname*{inj\,lim}_i \operatorname{Hom}_{\operatorname{Pro}\mathcal{D}}(F_i, F) \tag{B.11}$$

Below we will work in (full) subcategories of $\operatorname{Pro}\Delta^\circ\mathcal{P}$. It is clear that $\operatorname{Pro}\Delta^\circ\mathcal{P}$ is a simplicial category (it may be enriched in simplicial sets). The functors $-\times S$ and $\underline{\operatorname{Hom}}(S,-)$ for $S \in \Delta^\circ\,\mathrm{Set}$ may be extended to $\operatorname{Pro}\Delta^\circ\mathcal{P}$ and they remain adjoints. In particular cylinder and path objects exist in $\operatorname{Pro}\Delta^\circ\mathcal{P}$ and we may define homotopy fiber products and equalizers in $\operatorname{Pro}\Delta^\circ\mathcal{P}$.

It also clear that the Definition B.2.1 make sense in this context and furthermore we can define $\operatorname{Ho}\operatorname{Pro}\Delta^\circ\mathcal{P}$ using the formula (B.5).

B.6. **Pro-hypercoverings.** We consider the full subcategory $\operatorname{Pro}\mathcal{H}(\mathcal{P})$ of $\operatorname{Pro}\Delta^\circ\mathcal{P}$. We refer to the objects in $\operatorname{Pro}\mathcal{H}(\mathcal{P})$ as pro-hypercoverings.

We note the following generalization of Proposition B.4.3.

Proposition B.6.1. (1) *Let F be a pro-hypercovering and S a finite simplicial set (i.e. S has only a finite number of non-degenerate simplices). Then $\underline{\operatorname{Hom}}(S,F)$ is a pro-hypercovering. In particular the path object of F is a pro-hypercovering.*

(2) *Homotopy fiber products and homotopy equalizers of pro-hypercoverings are pro-hypercoverings.*

Proof. (1) follows directly from Proposition B.4.3(1) and (2) follows from Proposition B.4.3(1) and the fact that the diagrams for computing homotopy fiber products and equalizers can be constructed levelwise (see §B.5). □

We say that $F \in \operatorname{Pro}\mathcal{H}(\mathcal{P})$ is *homotopy projective* (with respect to $\mathcal{H}(\mathcal{P})$) if every diagram of solid arrows

$$\begin{array}{ccc} & & F \\ & \swarrow^{\cdots} & \downarrow \\ C' & \longrightarrow & C \end{array} \tag{B.12}$$

in $\operatorname{Pro}\mathcal{H}(\mathcal{P})$ with $C, C' \in \mathcal{H}(\mathcal{P})$ can be completed with a dotted arrow in $\operatorname{Pro}\mathcal{H}(\mathcal{P})$ such that the resulting diagram is commutative in $\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$. Let us denote the category of projective pro-hypercoverings by $\operatorname{Proj}\mathcal{H}(\mathcal{P})$.

The following is our main technical result.

Proposition B.6.2. *For every pro-hypercovering F there exists a map of pro-hypercoverings $F^* \to F$ such that F^* is homotopy projective.*

Proof. The proof is adapted from the proof of [5, Thm 2.7]. Let $\mathcal{P}_0 \subset \mathcal{P}$ be as in §B.3 but choose $\mathcal{P}_0$ large enough such that $F \in \operatorname{Pro} \mathcal{H}(\mathcal{P}_0)$ (up to isomorphism). This may be done by choosing the cardinal $\mathfrak{a}$ in Lemma B.3.2 large enough.

We will temporarily work in $\operatorname{Pro} \mathcal{H}(\mathcal{P}_0)$. We start by well-ordering the diagrams in $\operatorname{Pro} \mathcal{H}(\mathcal{P}_0)$

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ C' & \longrightarrow & C \end{array}$$

with C, $C' \in \mathcal{H}(\mathcal{P}_0)$.

We construct an ordinal sequence

$$\cdots \to F_\omega \to \cdots \to F_1 \to F_0 \tag{B.13}$$

in $\operatorname{Pro} \mathcal{H}(\mathcal{P}_0)$ as follows: $F_0 = F$; at a limit ordinal λ let $F_\lambda = \operatorname{proj\,lim}_{\mu<\lambda} F_\mu$. To define F_λ for a successor cardinal $\lambda = \mu + 1$ let $C' \to C \leftarrow F$ be the least diagram (if existing) for the well ordering such that the diagram of solid arrows

$$\begin{array}{ccc} F_\mu & \longrightarrow & F \\ \vdots & & \downarrow \\ C' & \longrightarrow & C \end{array}$$

cannot be completed with the dotted arrow.

Put $F_\lambda = C' \overset{h}{\times}_C F_\mu$. Then the resulting diagram

$$\begin{array}{ccccc} F_\lambda & \longrightarrow & F_\mu & \longrightarrow & F \\ & \searrow & & & \downarrow \\ & & C' & \longrightarrow & C \end{array}$$

is commutative in $\operatorname{Pro} \mathcal{H}(\mathcal{P}_0)$ up to a strict homotopy.

Since $\mathcal{H}(\mathcal{P}_0)$ is small it follows that this procedure has to stop for some ordinal σ. Put $F^\sharp = F_\sigma$ Then it follows that any diagram of solid arrows

$$\begin{array}{ccc} F^\sharp & \longrightarrow & F \\ \vdots & & \downarrow \\ C' & \longrightarrow & C \end{array}$$

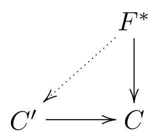
can be completed with the dotted arrow up to a strict homotopy.

Now define a sequence

$$\cdots \to F_1' \to F_0' = F^\sharp$$

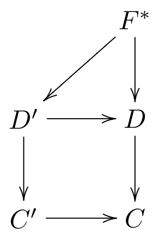
where $F'_{n+} = (F'_n)^\sharp$ and put $F^* = \operatorname{proj\,lim}_n F'_n$. We claim that any diagram of solid arrows

(B.14)



with C, $C' \in \mathcal{H}(\mathcal{P}_0)$ can be completed with the dotted arrow up to a strict homotopy. Indeed by (B.11) the vertical map is obtained from some map $F'_n \to C$. But then by construction we may factor F'_{n+1} through C'.

We now claim that F^* is homotopy projective. So we consider a solid diagram as in (B.14) but now we only require C, $C' \in \mathcal{H}(\mathcal{P})$. We have $F^* = \operatorname{proj\,lim}(F^*_\alpha)_{\alpha \in A}$ with $F^*_\alpha \in \mathcal{H}(\mathcal{P}_0)$. So the vertical map in (B.14) is obtained from some map $F^*_\alpha \to C$. Put $D = F^*_\alpha$. By Lemma B.4.7 we can construct a map of hypercoverings $D' \to C' \overset{h}{\times}_C D$ with $D' \in \mathcal{H}(\mathcal{P}_0)$. We may then construct a diagram in $\operatorname{Pro}\mathcal{H}(\mathcal{P})$



which is commutative in $\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$. This finishes the proof. □

Let $\mathcal{W}$ be the set of maps in $\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$ between homotopy projectives. We have the following result:

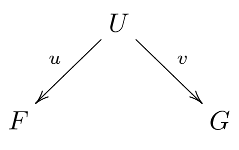
Proposition B.6.3. *For any $F, G \in \operatorname{Pro}\mathcal{H}(\mathcal{P})$ with F homotopy projective there is precisely one map $F \to G$ in $\mathcal{W}^{-1}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$.*

Proof.

Step 1. If $f, g : F \to G$ are maps in $\operatorname{Pro}\mathcal{H}(\mathcal{P})$ and F is homotopy projective then the images of f and g are the same in $\mathcal{W}^{-1}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$.

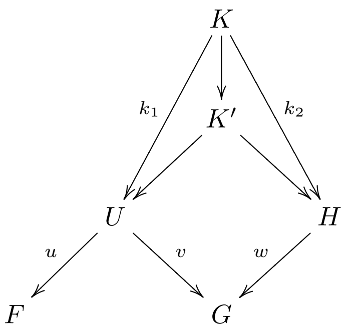
To see this let $K' \to F$ be the homotopy equalizer of f and g and let $K \to K'$ be a homotopy projective object mapping to K (constructed using B.6.2). If $k : K \to F$ is the composed map then fk and gk are the same in $\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$. Since k is in $\mathcal{W}$ this implies that f and g are the same in $\mathcal{W}^{-1}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$

Step 2. Any map $f : F \to G$ in $\mathcal{W}^{-1}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$ with F homotopy projective can be written as vu^{-1} where u, v fit in a diagram in $\operatorname{Pro}\mathcal{H}(\mathcal{P})$



with U homotopy projective.

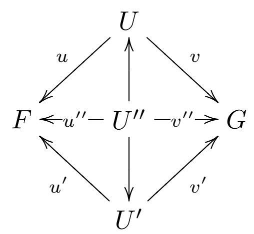
It is easy to see that it is sufficient to prove that if f is of the indicated form then so is $w^{-1}f$ with $w : H \to G$ in $\mathcal{W}$. To see this we make the following commutative diagram in $\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$.



where $K' = U \overset{h}{\times}_G H$ and K is homotopy projective. Then in $\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$ we have $wk_2 = vk_1$ and thus in $\mathcal{W}^{-1}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$: $w^{-1}v = k_2k_1^{-1}$. Hence $w^{-1}f = w^{-1}vu^{-1} = k_2(uk_1)^{-1}$.

Step 3. If $F, G \in \operatorname{Pro}\mathcal{H}(\mathcal{P})$ with F homotopy projective then there is at most one map $F \to G$ in $\mathcal{W}^{-1}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$.

Assume that there are two maps vu^{-1}, $v'u'^{-1}$ with "middle objects" U and U' as in Step 2. Let U'' be a homotopy projective mapping to $U \times U'$ (using Proposition B.6.2) . Using Step 1 we have a commutative diagram in $\mathcal{W}^{-1}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$



from which we obtain $vu^{-1} = v''u''^{-1} = v'u'^{-1}$.

Step 4. If $F, G \in \operatorname{Pro}\mathcal{H}(\mathcal{P})$ with F homotopy projective then there is precisely one map $F \to G$ in $\mathcal{W}^{-1}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$.

By Step 3 we only have to show that there is a map $F \to G$. Let K be a homotopy projective mapping to $F \times G$. Denote the maps of K to F and G by u and v. Then vu^{-1} is the required map. □

Let $\operatorname{Proj}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$ be the full subcategory of $\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$ consisting of homotopy projective objects. The same proof as the previous proposition, replacing $\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$ by $\operatorname{Proj}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$ yields the following result.

Corollary B.6.4. *The category $\mathcal{W}^{-1}\operatorname{Proj}\operatorname{Ho}\operatorname{Pro}\mathcal{H}(\mathcal{P})$ is equivalent to the singleton category. I.e. the category with one object and one (identity) arrow.*

B.7. **Complexes of sheaves of abelian groups.** For an abelian category $\mathcal{A}$ let $C(\mathcal{A})$ be the category of cochain complexes over $\mathcal{A}$.

Let us say that contravariant functor $H : \mathcal{P} \to \mathrm{Ab}$ is *weakly effaceable* if for every $G \in \mathcal{P}$ and for every $h \in H(G)$ there exists an epimorphism $\phi : F \to G$ in $\mathcal{P}$ such that $H(\phi)(h) = 0$.

We say that a contravariant functor $H : \mathcal{H}(\mathcal{P}) \to \mathrm{Ab}$ is *weakly effaceable* if for every $G \in \mathcal{H}(\mathcal{P})$ and for every $h \in H(G)$ there exists a map of hypercoverings $\phi : F \to G$ such that $H(\phi)(h) = 0$.

We will need the following result.

Lemma B.7.1. *Let $H : \mathcal{P} \to \mathrm{Ab}$ be a weakly effaceable functor which sends finite coproducts to products and let G be a hypercovering. Let $m \in \mathbb{Z}$ and let $a \in H(G_m)$. Then there exists a map of hypercoverings $\psi : F \to G$ such that $H(\psi_m)(a) = 0$ in $H(F_m)$.*

Proof. By Proposition B.4.5 we may assume that G is split. I.e.

$$G_m = \coprod_{\sigma:[m]\to[p] \text{ surj}} \sigma N_p(G)$$

and hence $a = \sum_{\sigma:[m]\to[p] \text{ surj}} \sigma a_\sigma$ where $a_\sigma \in H^p(N_p(G))$.

Let $N' \to N_p(G)$ is an epimorphism which effaces a_σ. Using Proposition B.4.6 we may refine G to a split hypercovering G' whose non-degenerate part is N' in degree p and which is unchanged in lower degrees.

Starting with the maximal p such that $a_p \neq 0$ and work our way down we eventually find a hypercovering in which the image of all a_p is zero. $\square$

Corollary B.7.2. *If $H : \mathcal{P} \to \mathrm{Ab}$ is weakly effaceable and sends finite coproducts to products then for all m the functor $\mathcal{H}(\mathcal{P}) \to \mathrm{Ab} : G \mapsto H(G_m)$ is effaceable as well.*

Lemma B.7.3. *For all acyclic complexes $\mathcal{L} \in C(\mathcal{P}_\mathbb{Z})$ the functor $H^0(\mathrm{Hom}(-, \mathcal{L})) : \mathcal{P} \to \mathrm{Ab}$ is weakly effaceable.*

Proof. Let $a \in H^0(\mathrm{Hom}(G, \mathcal{L}))$ with $G \in \mathcal{P}$. Thus a is represented by a map $G \to \ker(\mathcal{L}^0 \to \mathcal{L}^1) = \mathrm{im}(\mathcal{L}^{-1} \to \mathcal{L}^0)$. Let F be the pullback of the diagram

$$\begin{array}{ccc} & & \mathcal{L}^{-1} \\ & & \downarrow \\ G & \longrightarrow & \mathcal{L}^0 \end{array}$$

Then $F \to G$ is an epimorphism and the image of a in $H^0(\mathrm{Hom}(F, \mathcal{L}))$ is zero. $\square$

If A is a cosimplicial abelian group then as usual we denote by $C^*(A)$ the (unnormalized) cochain complex associated to A. If A is a cosimplicial object in the category of complexes of abelian groups then by $C^*(A)$ we will denote the total (product) complex of the double complex with rows $C^*(A^n)$.

If $F \in \Delta^\circ\mathcal{P}$ and $\mathcal{L}$ is a complex in $\mathcal{P}_\mathbb{Z}$ then by $\mathrm{Hom}(F, \mathcal{L})$ we denote the cosimplicial object in the category of complexes of abelian groups defined by

$$\mathrm{Hom}(F, \mathcal{L})^n = \mathrm{Hom}(F_n, \mathcal{L})$$

The following formula is clear

$$\underline{\mathrm{Hom}}(C_*(\mathbb{Z}F), \mathcal{L}) = C^*(\mathrm{Hom}(F, \mathcal{L})) \tag{B.15}$$

where the left hand side is the usual differentially graded Hom of complexes.

For $\mathcal{L} \in \mathcal{P}_{\mathbb{Z}}$ we define $H_{\mathcal{L}} : \mathcal{H}(\mathcal{P}) \to \mathrm{Ab}$ as the functor $H^0(C^*(\mathrm{Hom}(-, \mathcal{L})))$.

Lemma B.7.4. *If $\mathcal{L} \in \mathcal{P}_{\mathbb{Z}}$ is acyclic then the functor $H_{\mathcal{L}}$ is weakly effaceable.*

Proof. Assume $a \in H_{\mathcal{L}}(G)$ is represented by a morphism

$$a : C_*(\mathbb{Z}G) \to \mathcal{L}$$

Let $N_*(\mathbb{Z}G)$ be the normalized chain complex of G (i.e. the quotient of $C_*(\mathbb{Z}G)$ by the images of the degeneracies). It follows from the proof of [34, Thm 8.3.8] that the canonical map $C_*(\mathbb{Z}G) \to N_*(\mathbb{Z}G)$ is a homotopy equivalence. Indeed the proof shows that the kernel D of this map is of the form $\bigcup_p D_p$ such that D_{p+1}/D_p is contractible. Therefore D is itself contractible which is sufficient.

Hence up to homotopy we may view a as a map

$$b : N_*(\mathbb{Z}G) \to \mathcal{L}$$

Without loss of generality we may assume that G is split. Then $N_*(\mathbb{Z}G)_n = \mathbb{Z}N_n(G)$.

We must construct a map of hypercoverings $\phi : F \to G$ and a homotopy $h : N_*(\mathbb{Z}F) \to \mathcal{L}[-1]$ such that $b \circ \phi = dh + hd$.

We will construct F and h step by step. Suppose we have constructed a morphism of split hypercoverings $\phi' : F' \to G$ and maps $h'_i : \mathbb{Z}N_i(F') \to \mathcal{L}_{i+1}$ for $i < n$ such that $b'_i = dh'_i + h'_{i-1}d$ for $i = 0, \dots, n-1$ where $b' = b \circ \phi'$ and $h'_{-1} = 0$.

Put $c = b'_n - h'_{n-1}d$. Then $dc = 0$. Thus c defines an element $\bar{c}$ of $H^0(\mathrm{Hom}(N_n(F'), \mathcal{L}[-n]))$. By Lemma B.7.3 there exists an epimorphism $f : N' \to N_n(F')$ in $\mathcal{P}$ which effaces $\bar{c}$.

By Proposition B.4.6 there is a map of split hypercoverings $\psi : F'' \to F'$ such that $\psi_{\le n-1}$ is the identity and furthermore $N_n(F'') = N'$ in such a way that ψ_n restricts to the map f.

Put $h''_i = h'_i \circ \psi$, $b''_i = b'_i \circ \psi$. Then still $b''_i = dh''_i + h''_{i-1}d$ for $i < n$ but now $b''_n - h''_{n-1}d$ is of the form dh''_n for some map $h''_n : \mathbb{Z}N_n(F'') \to \mathcal{L}_{n+1}$. Repeating this procedure we ultimately construct the desired F and h. □

Below a contravariant functor $H : \mathcal{H}(\mathcal{P}) \to \mathrm{Ab}$ will be extended implicitly to a contravariant functor $\mathrm{Pro}\,\mathcal{H}(\mathcal{P}) \to \mathrm{Ab}$ by putting

$$H(\operatorname{proj\,lim}_{\alpha \in I} F_\alpha) = \operatorname{inj\,lim}_{\alpha \in I} H(F_\alpha)$$

Let us say that a contravariant functor $H : \mathcal{H}(\mathcal{P}) \to \mathrm{Ab}$ is *homotopy insensitive* if it factors through $\mathrm{Ho}\,\mathcal{H}(\mathcal{P})$. This is equivalent with demanding that H inverts constant path maps. Since this condition lifts to pro-objects it follows in particular that H extends to a functor $\mathrm{Ho}\,\mathrm{Pro}\,\mathcal{H}(\mathcal{P}) \to \mathrm{Ab}$.

Lemma B.7.5. *$H_{\mathcal{L}}(-)$ is homotopy insensitive for any $\mathcal{L} \in C(\mathcal{P}_{\mathbb{Z}})$.*

Proof. If $F \in \mathcal{H}(\mathcal{P})$ then according to Corollary B.2.3, the constant path map $F \to F^I$ is a combinatorial homotopy equivalence. It follows from Lemma B.2.5 that $\mathbb{Z}F \to \mathbb{Z}(F^I)$ is a combinatorial homotopy equivalence in $\Delta^\circ \mathcal{P}_{\mathbb{Z}}$.

Hence by Lemma B.2.4 the induced map $C_*(\mathbb{Z}F) \to C_*(\mathbb{Z}F^I)$ is a homotopy equivalence. It follows that $\underline{\mathrm{Hom}}(C_*(\mathbb{Z}F^I), \mathcal{L}) \to \underline{\mathrm{Hom}}(C_*(\mathbb{Z}F), \mathcal{L})$. is a homotopy equivalence. Then formula (B.15) finishes the proof. □

Lemma B.7.6. *Let $H : \mathcal{H}(\mathcal{P}) \to \mathrm{Ab}$ be a homotopy insensitive weakly effaceable functor. Then for any homotopy projective $F \in \mathrm{Ho}\,\mathrm{Pro}\,\mathcal{H}(\mathcal{P})$ we have $H(F) = 0$.*

Proof. We have $F = \operatorname{proj\,lim}_{\alpha\in A} F_\alpha$ with $F_\alpha \in \mathcal{H}(\mathcal{P})$ and $H(F) = \operatorname{inj\,lim}_\alpha H(F_\alpha)$. Let $h \in H(F)$. Then h is represented by some $h_\alpha \in H(F_\alpha)$. Since H is weakly effaceable there exists a map of hypercoverings $F' \to F_\alpha$ such that the image of h_α in $H(F')$ is zero. Since F is homotopy projective the map $F \to F_\alpha$ factors through F' in $\mathrm{Ho}\,\mathrm{Pro}\,\mathcal{H}(\mathcal{P})$. This implies that h is zero. □

If F is the pro-object $(F_\alpha)_\alpha$ then we define

$$C^*(\mathrm{Hom}(F, \mathcal{L}))_{\mathrm{pro}} = \operatorname*{inj\,lim}_\alpha C^*(\mathrm{Hom}(F_\alpha, \mathcal{L}))$$

and

$$C^*(\mathrm{Hom}(F, \mathcal{L})) = C^*(\operatorname*{inj\,lim}_\alpha \mathrm{Hom}(F_\alpha, \mathcal{L}))$$

We show below that $C^*(\mathrm{Hom}(F, \mathcal{L}))_{\mathrm{pro}}$ is well-behaved in its first argument and $C^*(\mathrm{Hom}(F, \mathcal{L}))$ is well behaved in its second argument. Furthermore there is an obvious map

$$C^*(\mathrm{Hom}(F, \mathcal{L}))_{\mathrm{pro}} \to C^*(\mathrm{Hom}(F, \mathcal{L}))$$

which is an isomorphism if $\mathcal{L}$ has left bounded cohomology (see Lemma B.7.9 below).

Proposition B.7.7. *If $f : F \to G$ is a map between homotopy projective pro-hypercoverings then $H_{\mathcal{L}}(f)$ is invertible for any $\mathcal{L} \in C(\mathcal{P}_{\mathbb{Z}})$.*

Proof. Note first that

$$H_{\mathcal{L}}(F) = H^0(C^*(\mathrm{Hom}(F, \mathcal{L})_{\mathrm{pro}}) \tag{B.16}$$

As $\mathcal{P}_{\mathbb{Z}}$ is a Grothendieck category there is a quasi-isomorphism $q : \mathcal{L} \to \mathcal{E}$ where $\mathcal{E}$ is homotopy injective [1]. I.e. $\underline{\mathrm{Hom}}_{\mathcal{P}_{\mathbb{Z}}}(\mathcal{Q}, \mathcal{E})$ is acyclic for every acyclic $\mathcal{Q}$.

Using Proposition B.4.2 and formula (B.15) we obtain that for any hypercovering E the canonical map

$$C^*(\mathrm{Hom}(E, \mathcal{E})) \to \mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(\underline{\mathbb{Z}}, \mathcal{E})$$

is a quasi-isomorphism. Taking direct limits we obtain that for any pro-hypercovering E we have a canonical quasi-isomorphism

$$C^*(\mathrm{Hom}(E, \mathcal{E}))_{\mathrm{pro}} \to \mathrm{Hom}_{\mathcal{P}_{\mathbb{Z}}}(\underline{\mathbb{Z}}, \mathcal{E}) \tag{B.17}$$

Let $\mathcal{C}$ be the cone of q. Then $\mathcal{C}$ is acyclic. We obtain a morphism of distinguished triangles in $K(\mathrm{Ab})$ (the homotopy category of Ab):

$$\begin{array}{ccccccc}
C^*(\mathrm{Hom}(F, \mathcal{L}))_{\mathrm{pro}} & \longrightarrow & C^*(\mathrm{Hom}(F, \mathcal{E}))_{\mathrm{pro}} & \longrightarrow & C^*(\mathrm{Hom}(F, \mathcal{C}))_{\mathrm{pro}} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
C^*(\mathrm{Hom}(G, \mathcal{L}))_{\mathrm{pro}} & \longrightarrow & C^*(\mathrm{Hom}(G, \mathcal{E}))_{\mathrm{pro}} & \longrightarrow & C^*(\mathrm{Hom}(G, \mathcal{C}))_{\mathrm{pro}} & \longrightarrow &
\end{array}$$

By Lemmas B.7.4,B.7.5,B.7.6 and (B.16) $C^*(\mathrm{Hom}(F, \mathcal{C}))_{\mathrm{pro}}$ and $C^*(\mathrm{Hom}(G, \mathcal{C}))_{\mathrm{pro}}$ are acyclic. By (B.17) the middle vertical map is a quasi-isomorphism. Hence it follows that the left most vertical map is a quasi-isomorphism as well. □

Lemma B.7.8. *Assume that $F \in \mathrm{Pro}\,\mathcal{H}(\mathcal{P}_{\mathbb{Z}}))$ is homotopy projective. If $\mathcal{L} \in C(\mathcal{P}_{\mathbb{Z}})$ is acyclic then we have that $C^*(\mathrm{Hom}(F, \mathcal{L}))$ is acyclic.*

Proof. Let the index of $\mathcal{L}$ be denoted by $q \in \mathbb{Z}$ and let the index of an object in $\mathcal{H}(\mathcal{P})$ be denoted by $p \in \mathbb{N}$. Let H be the functor $\mathcal{H}(\mathcal{P}) \to \mathrm{Ab}$ given by

$$H(G) = \bigoplus_{p,q} H^p(H^q(\mathrm{Hom}(G, \mathcal{L})))$$

By Lemma B.7.3 and Corollary B.7.2 we see that $U^q = H^q(\mathrm{Hom}(-, \mathcal{L}))$ is weakly effaceable. Thus $U = \oplus_q U^q$ is weakly effaceable as well. By an argument similar to Lemma B.7.5 we deduce that $H^p(U)$ is homotopy insensitive. Thus we conclude that H is both weakly effaceable and homotopy insensitive. By Lemma B.7.6 we conclude $H(F) = 0$ for $F \in \mathcal{H}(\mathcal{P})$ and hence for $F \in \mathrm{Pro}\,\mathcal{H}(\mathcal{P})$.

Hence the E_2 term of the spectral sequence computing the cohomology of $C^*(\mathrm{Hom}(F, \mathcal{L}))$ vanishes. Either by invoking the correct convergence criterion or by a direct diagram chase (which the author did) this implies that $C^*(\mathrm{Hom}(F, \mathcal{L}))$ is acyclic. □

Let $C^+(\mathcal{P}_\mathbb{Z})$ be the full subcategory of $C(\mathcal{P}_\mathbb{Z})$ consisting of complexes with left bounded cohomology.

Lemma B.7.9. *Assume that $\mathcal{L} \in C^+(\mathcal{P}_\mathbb{Z})$. Then the canonical map*

$$C^*(\mathrm{Hom}(F, \mathcal{L}))_{\mathrm{pro}} \to C^*(\mathrm{Hom}(F, \mathcal{L}))$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{L} \to \mathcal{E}$ to a left bounded complex of injectives $\mathcal{E}$. Let $\mathcal{C}$ be the cone. Then we have the following morphism of distinguished triangles in $K(\mathrm{Ab})$

$$\begin{array}{ccccccc} C^*(\mathrm{Hom}(F, \mathcal{L}))_{\mathrm{pro}} & \longrightarrow & C^*(\mathrm{Hom}(F, \mathcal{E}))_{\mathrm{pro}} & \longrightarrow & C^*(\mathrm{Hom}(F, \mathcal{C}))_{\mathrm{pro}} & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ C^*(\mathrm{Hom}(F, \mathcal{L})) & \longrightarrow & C^*(\mathrm{Hom}(F, \mathcal{E})) & \longrightarrow & C^*(\mathrm{Hom}(F, \mathcal{C})) & \longrightarrow & \end{array}$$

By Lemmas B.7.5,B.7.4,B.7.6 and Lemma B.7.8 $C^*(\mathrm{Hom}(F, \mathcal{C}))_{\mathrm{pro}}$ and $C^*(\mathrm{Hom}(F, \mathcal{C}))$ are acyclic. Furthermore since $\mathcal{E}$ is left bounded it is easy to see that the middle map is an isomorphism. Hence the left most map is a quasi-isomorphism. □

For a homotopy projective F in $\mathrm{Pro}\,\mathcal{H}(\mathcal{P})$ let Π_F be the functor

$$\Pi_F : C(\mathcal{P}_\mathbb{Z}) \to \Delta C(\mathrm{Ab}) : \mathcal{L} \mapsto \mathrm{Hom}(F, \mathcal{L})$$

Lemma B.7.10. *The functor $C^* \circ \Pi_F$ sends weak equivalences to quasi-isomorphisms.*

Proof. By considering the cones of quasi-isomorphisms, it is sufficient to prove that for any acyclic $\mathcal{L} \in C(\mathcal{P}_\mathbb{Z})$ we have that $C^*(\mathrm{Hom}(F, \mathcal{L}))$ is acyclic. This is precisely Lemma B.7.8. □

The following proposition is the raison d'être for the functor Π_F.

Lemma B.7.11. *When restricted to $C^+(\mathcal{P}_\mathbb{Z})$ the composition $C^* \circ \Pi_F$ is canonically isomorphic to* $\mathrm{RHom}_{\mathcal{P}_\mathbb{Z}}(\underline{\mathbb{Z}}, -)$.

Proof. If $\mathcal{E}$ is a left bounded complex of injectives then

$$C^*(\mathrm{Hom}(F, \mathcal{E})) = C^*(\mathrm{Hom}(F, \mathcal{E}))_{\mathrm{pro}}$$

and the latter is equal to $\mathrm{Hom}_{\mathcal{P}_\mathbb{Z}}(\underline{\mathbb{Z}}, \mathcal{E})$ by (B.17) □

B.8. **Sheaves of algebras.** In addition to the above notations, in this section $\mathcal{O}(n)_n$ will be a fixed DG-operad of abelian groups. We write $\mathrm{Alg}(\mathcal{O})$ for the category of $\mathcal{O}$-algebras.

Since by Lemma B.3.3 $\mathcal{P}_{\mathbb{Z}}$ is a symmetric monoidal category we may speak of $\mathcal{O}$-algebra objects on $\mathcal{P}$. We define $\mathrm{Alg}(\mathcal{P},\mathcal{O})$ as the category of $\mathcal{O}$-algebras in $\mathcal{P}$.

We make the following definitions.

(1) A weak equivalence in $\mathrm{Alg}(\mathcal{P},\mathcal{O})$ is a quasi-isomorphism.
(2) A map $A \to B$ in $\Delta\,\mathrm{Alg}(\mathcal{O})$ is a weak equivalence if $C^*(A) \to C^*(B)$ is a quasi-isomorphism.

Following custom the classes of weak equivalences will be denoted by $\mathcal{W}$. Note that if $F \in \mathcal{P}$ and $\mathcal{A} \in \mathrm{Alg}(\mathcal{P},\mathcal{O})$ then by construction $\mathrm{Hom}(F,\mathcal{A}) \in \Delta\,\mathrm{Alg}(\mathcal{O})$. Let Π_* be the bifunctor

$$\Pi_* : \mathrm{Pro}\,\mathcal{H}(\mathcal{P}) \times \mathrm{Alg}(\mathcal{P},\mathcal{O}) \to \Delta\,\mathrm{Alg}(\mathcal{O}) : \mathcal{A} \mapsto \mathrm{Hom}(F,\mathcal{A})$$

Let $\mathcal{W}^{\mathrm{cp}}$ be the constant path maps in $\mathrm{Pro}\,\mathcal{H}(\mathcal{P})$. Let $\mathrm{Alg}^+(\mathcal{P},\mathcal{O})$ be the full subcategory of $\mathrm{Alg}(\mathcal{P},\mathcal{O})$ whose objects have left bounded cohomology.

According to Lemma B.7.5 and Lemma B.7.10 we obtain a bifunctor

$$\Pi_* : \mathcal{W}^{\mathrm{cp},-1}\,\mathrm{Pro}\,\mathcal{H}(\mathcal{P}) \times \mathrm{Alg}(\mathcal{P},\mathcal{O}) \to \mathcal{W}^{-1}\Delta\,\mathrm{Alg}(\mathcal{O}) : \mathcal{A} \mapsto \mathrm{Hom}(F,\mathcal{A})$$

and hence a bifunctor

$$\Pi_* : \mathrm{Ho}\,\mathrm{Pro}\,\mathcal{H}(\mathcal{P}) \times \mathrm{Alg}(\mathcal{P},\mathcal{O}) \to \mathcal{W}^{-1}\Delta\,\mathrm{Alg}(\mathcal{O}) : \mathcal{A} \mapsto \mathrm{Hom}(F,\mathcal{A})$$

Π_* restricts to a bifunctor

$$\Pi_* : \mathrm{Proj}\,\mathrm{Ho}\,\mathrm{Pro}\,\mathcal{H}(\mathcal{P}) \times \mathrm{Alg}(\mathcal{P},\mathcal{O}) \to \mathcal{W}^{-1}\Delta\,\mathrm{Alg}(\mathcal{O}) : \mathcal{A} \mapsto \mathrm{Hom}(F,\mathcal{A})$$

Using Proposition B.7.7 we obtain a bifunctor

$$\Pi_* : \mathcal{W}^{-1}\,\mathrm{Proj}\,\mathrm{Ho}\,\mathrm{Pro}\,\mathcal{H}(\mathcal{P}) \times \mathrm{Alg}(\mathcal{P},\mathcal{O}) \to \mathcal{W}^{-1}\Delta\,\mathrm{Alg}(\mathcal{O}) : \mathcal{A} \mapsto \mathrm{Hom}(F,\mathcal{A})$$

By Corollary B.6.4 the first argument of Π_* is now a singleton category.

Below we define $\Sigma = \Pi_*(F,-)$ for an arbitrary pro-hypercovering F. It follows from the above discussion that Σ is well defined up to a unique natural isomorphism. It follows from Lemma B.7.11 that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Alg}^+(\mathcal{P},\mathcal{O}) & \xrightarrow{\Sigma} & \Delta\,\mathrm{Alg}(\mathcal{O}) \\ \downarrow & & \downarrow C^* \\ C(\mathcal{P}_{\mathbb{Z}}) & \xrightarrow[\mathrm{RHom}_{\mathcal{P}_{\mathbb{Z}}}(\underline{\mathbb{Z}},-)]{} & D(\mathrm{Ab}) \end{array}$$

where the left arrow is the forgetful functor.

Let $F \in \mathrm{Pro}\,\mathcal{H}(\mathcal{P})$ and $\mathcal{A} \in \mathrm{Alg}^+(\mathcal{O})$. Choose an arbitrary projective pro-hypercovering P. According to Proposition B.6.3 there is a unique map $P \to F$ in $\mathcal{W}^{-1}\,\mathrm{Pro}\,\mathcal{H}(\mathcal{P})$. In this way we obtain a canonical map

$$\mathrm{Hom}(F,\mathcal{A}) \to \mathrm{Hom}(P,\mathcal{A}) \cong \Sigma\mathcal{A} \tag{B.18}$$

in $\mathcal{W}^{-1}\Delta\,\mathrm{Alg}(\mathcal{O})$.

Proposition B.8.1. *Assume* $\mathcal{A} \in \mathrm{Alg}^+(\mathcal{P},\mathcal{O})$ *and* $F \in \mathcal{H}(\mathcal{P})$ *is such that* $\mathrm{Ext}^i_{\mathcal{P}_{\mathbb{Z}}}(\mathbb{Z}F_m, \mathcal{A}^n) = 0$ *for all* $m \geq 0$, $n \in \mathbb{Z}$ *and* $i > 0$. *Then* (B.18) *is an isomorphism in* $\mathcal{W}^{-1}\Delta\,\mathrm{Alg}(\mathcal{O})$.

Proof. We have to show that

$$C^*(\mathrm{Hom}(F,\mathcal{A})) \to C^*(\mathrm{Hom}(P,\mathcal{A}))$$

is a quasi-isomorphism. By formula (B.15) and our vanishing hypotheses we obtain $C^*(\mathrm{Hom}(F,\mathcal{A})) \cong \mathrm{RHom}_{\mathcal{P}_\mathbb{Z}}(\underline{\mathbb{Z}},\mathcal{A})$. It follows from Lemma B.7.11 that $C^*(\mathrm{Hom}(P,\mathcal{A})) \cong \mathrm{RHom}_{\mathcal{P}_\mathbb{Z}}(\underline{\mathbb{Z}},\mathcal{A})$ as well. □

B.9. **Čech cohomology.** In this section we discuss the important special case of Čech cohomology. Let X be a topological space and let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of X. As usual we identify $U \in \mathrm{Open}(X)$ with the representable sheaf $\mathrm{Hom}_{\mathrm{Open}(X)}(-,U)$. Then the unordered Čech covering of X is the simplicial sheaf on X which in degree m is equal to

$$C(\mathcal{U})_m = \coprod_{i_0,\ldots,i_m} U_{i_0} \cap \cdots \cap U_{i_m}$$

It is well-known and easy to see that this a hypercovering.

If given an ordering on I we may also define

$$C^o(\mathcal{U})_m = \coprod_{i_0 \le \cdots \le i_m} U_{i_0} \cap \cdots \cap U_{i_m}$$

Note that the inclusion map

$$C^o(\mathcal{U}) \to C(\mathcal{U})$$

is a map of simplicial sheaves.

Let $\mathcal{A} \in \mathrm{Alg}^+(\mathcal{P},\mathcal{O})$. The unordered and ordered Čech complexes of $\mathcal{A}$ are respectively defined as the cosimplicial complexes of $\mathcal{O}$-algebras

$$\begin{aligned} \mathrm{Ch}(\mathcal{U},\mathcal{A}) &= \mathrm{Hom}(C(\mathcal{U}),\mathcal{A}) \\ \mathrm{Ch}^o(\mathcal{U},\mathcal{A}) &= \mathrm{Hom}(C^o(\mathcal{U}),\mathcal{A}) \end{aligned} \tag{B.19}$$

Lemma B.9.1. *Assume that for all $m \ge 0$, $\{i_0,\ldots,i_m\} \subset I$, $j > 0$ and $n \in \mathbb{Z}$ we have*

$$H^i(U_{i_0} \cap \cdots \cap U_{i_m}, \mathcal{A}^n) = 0$$

Then

$$\mathrm{Ch}(\mathcal{U},\mathcal{A}) \cong \mathrm{Ch}^o(\mathcal{U},\mathcal{A}) \cong \Sigma(\mathcal{A})$$

in $\mathcal{W}^{-1}C^+(\mathcal{P}_\mathbb{Z})$.

Proof. Since $\mathrm{Ch}(\mathcal{U},\mathcal{A})$ is a hypercovering the isomorphism $\mathrm{Ch}(\mathcal{U},\mathcal{A}) \cong \Sigma(\mathcal{A})$ follows from Proposition B.8.1.

The ordered Čech covering is not a hypercovering but nevertheless, by looking at stalks, it is easy to see that $C^*(\mathbb{Z}C^o(\mathcal{U}))$ is a resolution of the constant sheaf $\mathbb{Z}_X$. From this we deduce that source and target of the map

$$C^*(\mathrm{Ch}^o(\mathcal{U},\mathcal{A})) \to \mathrm{Ch}(\mathcal{U},\mathcal{A})$$

compute $\mathrm{RHom}(\mathbb{Z}_X,\mathcal{A})$. Hence it is a quasi-isomorphism. □

B.10. **Relation to Hinich's construction.** We now assume that $\mathcal{P} = \mathrm{Sh}(\mathcal{C})$ for a small site $\mathcal{C}$. A presheaf over $\mathcal{C}$ is said to be *semi-representable* if it is a coproduct of representable presheaves. We will say that a simplicial presheaf F is a presheaf-hypercovering if the associated simplicial sheaf aF is a hypercovering in the above sense. We will say that F is a Verdier hypercovering if each F_n is semi-representable. We denote the corresponding categories by $\mathcal{H}^{\mathrm{Pre}}(\mathcal{C})$ and $\mathcal{H}^{\mathrm{V}}(\mathcal{C})$.

If $U \in \mathcal{C}$ then a simplicial presheaf F with an augmentation $F \to U$ will be called a Verdier-hypercovering of U if F is a Verdier-hypercovering of U in the site $\mathcal{C}/U$.

Following [21] we say that a complex of presheaves is fibrant if if for any $U \in \mathcal{C}$ and for any Verdier-hypercovering $F \to U$ we have that $M(U) \to C^*(\mathrm{Hom}(F, M))$ is a quasi-isomorphism.

Hinich proves under some hypotheses on $\mathcal{O}$ (which hold if $\mathcal{O}$ is k-linear over a field of characterstic zero) that for any presheaf of $\mathcal{O}$-algebras $\mathcal{A}$ there is a map of presheaves of $\mathcal{O}$-algebras $\mathcal{A} \to \mathcal{A}'$ with $\mathcal{A}'$ fibrant which is a quasi-isomorphism after sheaffication. The derived global sections of $\mathcal{A}$ are then given by $\mathrm{Hom}(F, \mathcal{A}')^{TS}$ for a Verdier hypercovering F of e. If e itself is in $\mathcal{C}$ then we may consider it as its own hypercovering and in this case we may dispense with the Thom-Sullivan normalization. I.e. we may define the derived global sections of $\mathcal{A}$ as $\mathcal{A}'(e)$.

We will show that in case $\mathcal{A}$ is a sheaf of $\mathcal{O}$-algebras with left bounded grading this yields the same result as our construction. Mimicking the above methods we may produce a pro-object $P = (P_\alpha)_\alpha$ in $\mathcal{H}^V(\mathcal{C})$ mapping to the hypercovering F such that any diagram of solid arrows

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ H_1 & \longrightarrow & H_2 \end{array}$$

with H_1, H_2 in $\mathcal{H}^{\mathrm{Pre}}(\mathcal{C})$ can be factored like the dotted arrow. It follows in particular that aP is a homotopy projective object in $\mathrm{Pro}\,\mathcal{H}(\mathcal{P})$. Hence we need to prove that $\mathrm{Hom}(P, \mathcal{A})$ is weaky equivalent to $\mathrm{Hom}(F, \mathcal{A}')$.

We have now maps

$$\mathrm{Hom}(P, \mathcal{A}) \to \mathrm{Hom}(P, \mathcal{A}') \leftarrow \mathrm{Hom}(F, \mathcal{A}')$$

and it it is sufficient prove that these are weak equivalences. By a suitable analogue of Lemma B.7.10 the first maps is a weak equivalence. By an analogue of Lemma B.7.9 we have that $\mathrm{Hom}(P, \mathcal{A}')$ is weakly equivalent to $\operatorname{proj\,lim}_\alpha \mathrm{Hom}(P_\alpha, \mathcal{A}')$. Hence it is sufficient to show that

$$\mathrm{Hom}(P_\alpha, \mathcal{A}') \leftarrow \mathrm{Hom}(F, \mathcal{A}')$$

is a weak equivalence. This follows from the fact that $\mathrm{Hom}(F, \mathcal{A}')$ is, up to weak equivalence, independent of the Verdier hypercovering F. See [21, §1.4.3].

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Departement WNI, Universiteit Hasselt, Universitaire Campus, Building D, 3590 Diepenbeek, Belgium

E-mail address, M. Van den Bergh: michel.vandenbergh@uhasselt.be