# TRACE RINGS OF GENERIC MATRICES ARE COHEN-MACAULAY.

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Abstract. In this paper we prove that trace rings of generic matrices are Cohen-Macaulay (7.3.6). This is done by relating this problem to a conjecture of Stanley about modules of invariants under a reductive group.

We prove a slightly weakened version (3.4') of this conjecture in special cases (6.1.9). In particular we obtain that (3.4') is true for $SL_2$ (6.1.11).

## 1. Introduction.

Let $G$ be a reductive algebraic group over an algebraically closed field of characteristic zero and let $W$ be a finite dimensional representation of $G$. Then $G$ acts on the polynomial ring $k[W]$ and the Hochster Roberts theorem [8] tells us that $k[W]^G$ is Cohen-Macaulay.

A first objective in this paper will be to study a situation that looks very similar. Let $U$ be another finite dimensional $G$-representation. Then $U \otimes_k k[W]$ is a free $k[W]$-module and a natural generalization of the Hochster Roberts theorem would be that $(U \otimes_k k[W])^G$ is a Cohen Macaulay $k[W]^G$-module.

Unfortunately, it is easy to see that this cannot be true in general (see Ex. 3.1). There is however a conjecture, due to Stanley [20], that gives at least some cases under which the above statement is true.

We will not say anything about Stanley's original conjecture. Instead we will replace it with a slightly weaker version (Conj. 3.4').

The first main result in this paper is that we prove Conj. 3.4' for certain pairs $(G, W)$. Namely if $X = \text{Spec}k[W]$ and $X^u$ is the locus of the $G$-unstable points in $X$ then we require that $X^u$ should be constructible, i.e. that it can be build up from smaller managable parts in a sense explained in section 6. As a corollary we immediately obtain that Conj. 3.4' holds if $G = SL_2$ (6.1.11).

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Another situation that can be handled by the methods developed in this paper is

$$G = SL(V) \text{ and } W = \text{End}(V)^{m*} \tag{1}$$

In the last section of this paper we will show that in this case $X^u$ is constructible and hence Conj. 3.4' holds.

Our main motivation for studying the situation (1) lies in our interest in the trace rings of generic matrices. Let $M_n$ be the variety of $n \times n$-matrices. $(M_n)^m$ will be the $m$-fold product (over Spec$k$) $M_n \times M_n \times \cdots \times M_n$. Let $G = SL_n$. Then one defines

$$\mathbb{T}_{m,n} = \{f : (M_n)^m \to M_n \mid f \text{ polynomial and } G\text{-equivariant}\} \tag{2}$$

$\mathbb{T}_{m,n}$ is a non-commutative ring (using the multiplication in $M_n$) and its center is given by

$$Z_{m,n} = \{f : (M_n)^m \to \text{Spec}k \mid f \text{ polynomial and } G\text{-equivariant}\}$$

$Z_{m,n}$ is the commutative and $\mathbb{T}_{m,n}$ is the non-commutative trace ring of $m$ generic $n \times n$-matrices. They were first extensively studied by M. Artin and C. Procesi. M. Artin proved that the maximal ideals of $Z_{m,n}$ parametrize semisimple representations of dimension $n$ of the free algebra $k\langle X_1, \dots, X_m\rangle$ and the two-sided maximal ideals of $\mathbb{T}_{m,n}$ correspond to the simple components of such representations [1][2].

Let $X_i$ be the projection of $(M_n)^m$ onto the $i$'th factor, and let $\text{Tr}(X_{i_1} \cdots X_{i_u}) : (M_n)^m \to \text{Spec}k$ be the corresponding trace maps. C. Procesi proved Artin's conjecture that $Z_{m,n}$ is generated over $k$ by the trace monomials $\text{Tr}(X_{i_1} \cdots X_{i_u})$ and $\mathbb{T}_{m,n}$ is generated as a module over $Z_{m,n}$ by the monomials in the $X_i$'s [16]. Furthermore he proved that all the relations between those generators can be obtained from the Cayley Hamilton polynomial (explaining the terminology of trace rings).

From this one easily deduces that $\mathbb{T}_{m,n}$ is a generic object in the category of $k$-algebras with a trace map. To be more precise, let $\Lambda$ be a $k$-algebra, equipped with a further unary operation $T : \Lambda \to \Lambda$, called trace, satisfying the list of conditions in [17, p. 194]. Assume furthermore that $T$ satisfies the Cayley Hamilton identities of $n \times n$-matrices. Let $a_1, \dots, a_m \in \Lambda$. Then there exists a unique map $\mathbb{T}_{m,n} \to \Lambda$ commuting with trace and sending $X_i$ to $a_i$.

If $n = 1$ then $\mathbb{T}_{m,n}$ is a polynomial ring and hence it has finite global dimension. A first natural question would be whether $\mathbb{T}_{m,n}$ *always* has finite global dimension (being a generic object). However the complete, somewhat disappointing result is given below.

**Theorem 1.1.** [5][11][12][18] $\mathbb{T}_{m,n}$ *has finite global dimension if and only if* $n = 1$, $m = 1$, $(m, n) = (2, 2)$, $(m, n) = (3, 2)$ *or* $(m, n) = (2, 3)$.

After computations in low dimensions, L. le Bruyn conjectured that $\mathbb{T}_{m,n}$ is always a Cohen-Macaulay $Z_{m,n}$-module. This was proved by him in the case of $2 \times 2$-matrices. Later he and C. Procesi proved that $(\mathbb{T}_{m,n})_p$ is Cohen-Macaulay if $p \in \mathrm{Max} Z_{m,n}$ corresponds to a semisimple representation of $k\langle X_1, \ldots, X_m\rangle$ having distinct irreducible components [11].

As the title of this paper indicates, we will prove that $\mathbb{T}_{m,n}$ is Cohen-Macaulay in general (7.3.6). This is done as follows. From (2) it is clear that

(3)
$$\mathbb{T}_{m,n} = (U \otimes k[W])^G \text{ where } G = SL(V), W = \mathrm{End}(V)^{m*}, U = \mathrm{End}(V)$$

Hence we are in the situation of (1) and Conj. 3.4' applies. Since it is easy to see that hypothesis of Conj. 3.4' are satisfied in this case, we are done.

This paper is organized as follows :

In section 2 we introduce some often used notations.

In section 3 we review Stanley's conjecture and we introduce the weaker version Conj. 3.4'.

In section 4 we outline our method for verifying the Cohen-Macaulayness of $(U \otimes k[W])^G$. I.e. we relate this problem to the cohomology modules $H^{\cdot}_{X^u}(X, \mathcal{O}_X)$.

In section 5 we introduce a spectral sequence (5.1) that may be interesting in its own right. It allows us, in some cases, to estimate the cohomology modules, introduced above.

In section 6 we try to break up $X^u$ into managable pieces that can be handled by the main result of section 5. This leads us to the notion of constructibility. We prove Conj. 3.4' for $G = SL_2$ (6.1.11). In the last subsection we introduce a combinatorial method for verifying constructibility.

Finally in section 7 we use the combinatorial criterion, derived in section 6, to show that $X^u$ is constructible in the situation (1). We obtain that $\mathbb{T}_{m,n}$ is Cohen-Macaulay in general (7.3.6).

## 2. Notations and conventions.

In the sequel $k$ will always be an algebraically closed field of characteristic zero.

If $G$ is a linear algebraic group over $k$ then $\mathcal{W}_G$ will be the Weylgroup of $G$. $Y(G)$ will be the pointed set of one-parameter subgroups of $G$.

An irreducible representation of $G$ defines a character $G \to G_m$. This is a polynomial map, invariant under conjugation (we will always assume that characters are characters of irreducible representations). If $T$ is a torus then the character of $T$ are homomophisms and they form an abelian group in the usual way. This group will be denoted by $X(T)$. and the group law will be written additively. We define $X(T)_{\mathbb{Q}}$ as $\mathbb{Q} \otimes_{\mathbb{Z}} X(T)$. Since $T$ is a torus, $Y(T)$ also carries an abelian group structure and there is a natural pairing $Y(T) \times X(T) \to X(G_m) \cong \mathbb{Z}$ given by composition. This pairing will be denoted by $\langle \ , \ \rangle$.

Characters of $T$ will be identified with one-dimensional representations of $T$. Hence the notation $\chi_1 \oplus \chi_2$ for $\chi_{1,2} \in X(T)$ stands for the two-dimensional representation of $T$ which is the direct sum of the one-dimensional representations determined by $\chi_1$ and $\chi_2$. This is not to be confused with $\chi_1 + \chi_2$ which is just the sum of $\chi_1$ and $\chi_2$ in $X(T)$.

If $P \subset G$ is an algebraic subgroup of $G$ and $X$ is a scheme with a $P$-action then $G \times^P X = G \times X/P$. There is a natural projection map $G \times^P X \to G/P$ given by $\overline{(g,x)} \mapsto \overline{g}$, with fibers isomorphic to $X$. Taking the fiber over $[P]$ in $G/P$ induces an equivalence between the category of quasicoherent $\mathcal{O}_{G \times^P X}$-modules with a $G$-action and the category of quasicoherent $\mathcal{O}_X$-modules with a $P$-action. The inverse of this equivalence will be denoted by $\tilde{}$.

Let $R$ be a $\mathbb{Z}$-graded Noetherian commutative ring of the form $k \oplus R_1 \oplus R_2 \oplus \cdots$ and let $M$ be a finitely generated graded $R$-module. The Poincare series of $M$ will be defined as

$$P(M,t) = \sum_{n=-\infty}^{+\infty} \dim(M_n) t^n$$

When we say that $M$ is Cohen-Macaulay, we always mean that $M$ is maximal Cohen-Macaulay. This is equivalent with the fact that $R$ contains a graded polynomial ring $R'$ over $k$ such $M$ is a finitely generated free $R'$-module.

## 3. A conjecture of Stanley.

In this section we will discuss a natural generalisation for the Hochster Roberts theorem on Cohen-Macaulayness of invariant rings. Unfortunately this generalisation is not true in general. There exists however a conjecture, due to Stanley [20], which gives at least some cases under which the generalisation is true.

Let $G$ be a reductive group over $k$ and let $U$, $W$ be two finite dimensional representations of $G$. Define $R = k[W]$, $d = \dim W$ and

$h = \dim R^G$. Then $G$ acts in a natural way on the free $R$-module $U \otimes_k R$.

By the Hochster Roberts theorem [8], $R^G$ is Cohen-Macaulay. It is therefore natural to ask whether $(U \otimes_k R)^G$ is a Cohen-Macaulay $R^G$-module. This is not always true however. Here is a simple counter example :

**Example 3.1.** Let $G = T = G_m$ and let $\chi$ be a generator for $X(T)$. Define $U = \chi^{-1}$ and $W = \chi \oplus \chi \oplus \chi^{-1}$. Then $R = k[x, y, z]$, $M = k[x, y, z]$ and $G_m$ acts on $R$ and $M$ as follows : let $\alpha \in G_m$, $f \in R$ and $g \in M$. Then $\alpha.f = f(\alpha x, \alpha y, \alpha^{-1} z)$ and $\alpha.g = \alpha^{-1} g(\alpha x, \alpha y, \alpha^{-1} z)$. Hence $R^G = k[xz, yz]$ and $M^G = (xz, yz)z^{-1}$. Clearly $M^G$ is not a Cohen-Macaulay $R^G$-module.

It is no restriction to assume that $U$ is irreducible because if $U = U_1 \oplus \cdots \oplus U_u$ then clearly $(U \otimes R)^G = (U_1 \otimes R) \oplus \cdots \oplus (U_u \otimes R)^G$. Hence from now on we will make this assumption. In that case $U^*$ is determined by its character $\chi : G \to G_m$.

For an arbitrary character of $G$, Stanley defines $R^G_\chi$ as the sum of all irreducible subrepresentations of $G$ with character $\chi$ [20]. Clearly $R = \oplus_\chi R^G_\chi$ where $\chi$ runs through all characters of $G$. The proof that $R^G$ is finitely generated also works for $R^G_\chi$ and since $R^G_\chi$ is obviously torsion free, one deduces that $\dim R^G_\chi = \dim R^G$ if $R^G_\chi \neq \emptyset$.

**Lemma 3.2.** *If $\chi$ is the character of $U^*$, then $R^G_\chi \cong U^* \otimes (U \otimes R)^G$.*

Hence the question whether $(U \otimes R)^G$ is Cohen-Macaulay is equivalent with the question whether $R^G_\chi$ is Cohen-Macaulay.

Assume now that $G = T$ is a torus, $\chi \in X(T)$ and let the weights of $W$ be given by $\alpha_1, \ldots, \alpha_d \in X(T)$. Then we say that $\chi$ is critical [20] for $(T, W)$ if the system $z_1\alpha_1 + \cdots + z_d\alpha_d = \chi$ in $X(T)_{\mathbb{Q}}$ has a rational solution $(a_1, \ldots, a_d)$ with the following properties :

- $a_i \leq 0$
- If $(b_1, \ldots, b_d)$ is an integer solution of $z_1\alpha_1 + \cdots + z_d\alpha_d = \chi$ such that $b_i \geq a_i$ then $b_i \geq 0$ for all $i$.

**Theorem 3.3.** [20] *Assume that $\chi$ is critical for $(T, W)$. Then $R^T_\chi$ is Cohen-Macaulay. Furthermore there is a functional equation*

$$P(R^T_\chi, 1/t) = (-1)^h t^d P(R^T_\psi, t) \tag{4}$$

*where $\psi = (\chi \det \chi)^*$. Here $*$ denotes the dual character and* $\det \chi$ *is the character of the highest exterior power of the representation corresponding to $\chi$.*

A character is clearly critical if it is of the form $\sum_{i=1}^{d} a_i\alpha_i$ where $-1 < a_i \leq 0$. We will call such a character strongly critical. This notion is useful because it is somewhat easier to check that a character is strongly critical than that it is critical.

Assume now that $G$ is arbitrary again and let $T \subset G$ be a maximal torus. Assume that $\chi : G \to G_m$ is a character. Then $\chi|T = \chi_1 \oplus \cdots \oplus \chi_u$ where $\chi_i \in X(T)$. Let $(\rho_1 \ldots, \rho_r)$ be the set of roots of $G$. Then Stanley [20] calls $\chi$ critical for $(G, W)$ if $\chi_i - \sum_{j\in S}\rho_j$ is critical for $(T, W)$ for all $1 \leq i \leq u$ and for all $S \subset \{1, \ldots, r\}$. He proves that if $\chi$ is critical then $R^G_\chi$ satisfies the functional equation (4). This leads to a natural conjecture :

**Conjecture 3.4.** If $\chi$ is critical for $(G, W)$ then $R^G_\chi$ is Cohen-Macaulay.

Of course a weaker version of this conjecture can be obtained if we require that all the $\chi_i - \sum_{j\in S}\rho_j$ are strongly critical. A character with this property will be called strongly critical for (G,W). In the sequel we will refer to the weaker version of Conj. 3.4 as Conj. 3.4'.

## 4. The method.

As in the previous section, $G$ will be a reductive algebraic group over $k$. $R = k[W]$ and $d = \dim R$, $h = \dim R^G$. $\chi$ will be some character of $G$. We define $I = R(R^G)^+$.

The following lemma will be basic in this paper.

**Lemma 4.1.** $H^{\cdot}_{(R^G)^+}(R^G_\chi) = H^{\cdot}_I(R)^G_\chi$. *(Here $H^{\cdot}_I(R)^G_\chi$ has the obvious meaning.)*

*Proof.* Let $f_1, \ldots, f_u$ be a set of generators for $(R^G)^+$. Then the $(f_i)_i$ are obviously also $R$ generators for $I$. Let $K^{\cdot}(R, f_1, \ldots, f_u)$ be the complex

$$0 \to \oplus_i R_{f_i} \to \oplus_{\substack{i,j\\ i<j}} R_{f_if_j} \to \cdots \to R_{f_1\cdots f_u} \to 0$$

with the standard boundary maps. Then $H^{\cdot}_I(R)^G_\chi = H^{\cdot}(K^{\cdot}(R, f_1, \ldots, f_u))^G_\chi$. But using the fact that $G$ is reductive, we easily deduce

$$\begin{aligned} H^{\cdot}(K^{\cdot}(R, f_1, \ldots, f_u))^G_\chi &= H^{\cdot}(K^{\cdot}(R, f_1, \ldots, f_u)^G_\chi) \\ &= H^{\cdot}(K^{\cdot}(R^G_\chi, f_1, \ldots, f_u)) \\ &= H^{\cdot}_{(R^G)^+}(R^G_\chi) \end{aligned}$$

□

**Corollary 4.2.** *$R^G_\chi$ is Cohen-Macaulay if and only if $H^i_I(R)^G_\chi = 0$ for $i = 0, \ldots, h-1$.*

*Proof.* This statement is vacuous and hence true if $R^G_\chi = 0$. So assume that $R^G_\chi \neq 0$. It is well known that $R^G_\chi$ is Cohen-Macaulay if and only if $H^i_{(R^G)^+}(R^G_\chi) = 0$ for $i = 0, \ldots, h-1$. Then the result follows from lemma 4.1. □

Let $X = \mathrm{Spec} k[W]$ and let $T$ be a maximal torus in $G$. The radical of the ideal $I$ is the defining ideal of the $G$-unstable locus in $X$ which will be denoted by $X^u$. I.e.

$$X^u = \{x \in X | 0 \in \overline{Gx}\}$$

$X^u$ maybe described more conveniently using the Hilbert Mumford criterion [14] which says that every point in $X^u$ is unstable for some one parameter subgroup of $G$. I.e. if $\lambda \in Y(G)$ then one defines

$$X_\lambda = \{x \in X | \lim_{t\to 0} \lambda(t)x = 0\}$$

and

$$G(\lambda) = \{g \in G | \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$$

Then $G(\lambda)$ acts on $X_\lambda$ and $G(\lambda)$ is a parabolic subgroup of $G$ [14, Prop. 2.6]. Then it follows from the Hilbert Mumford criterion that

$$X^u = \bigcup_{\lambda \in Y(T)} GX_\lambda \tag{5}$$

$GX_\lambda$ is the image of $G \times^{G(\lambda)} X_\lambda$ in $X$ under the canonical map. This map factors through the projection map $G/G(\lambda) \times X \to X$ and hence it is projective. Therefore its image is closed (this is a well known argument, see for example [13]).

Clearly $GX_\lambda = GX_{w(\lambda)}$ if $w \in \mathcal{W}_G$. Therefore one can restrict the union in (5) to a Weyl chamber in $Y(T)$. Let $B$ be a Borel subgroup of $G$ containing $T$. Then we have proved the following (well known) fact :

**Lemma 4.3.** *With notations and assumptions as above :*

$$X^u = \bigcup_{\substack{\lambda \in Y(T) \\ G(\lambda) \supset B}} GX_\lambda \tag{6}$$

## 5. A result on cohomology with support.

In lemma 3.2, we have seen that, to check Cohen-Macaulayness of modules of invariants, it is important to be able to compute cohomology with support in the unstable locus.

Our two main tools to handle this problem will be the standard long exact sequence for cohomology with support and Theorem 5.1 below.

**Theorem 5.1.** *Assume that we are in the following situation :*

$$\begin{array}{ccc} S' & \xrightarrow{j} & X' \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow[i]{} & X \end{array}$$

*where*

- *$X$, $X'$ are smooth over $k$ and $\pi$ is a smooth projective map.*
- *$i$ and $j$ are closed immersions.*
- *$\phi$ is the restriction of $\pi$ to $S'$ and it is settheoretically a bijection.*

*Let $\mathcal{M}$ be a quasicoherent sheaf on $X$. Then there is a spectral sequence :*

$$E_1^{pq} : H^q_{S'}(X', \pi^*\mathcal{M} \otimes \wedge^p \Omega_{X'/X}) \Rightarrow H_S^{p+q-2\dim(X'/X)}(X, \mathcal{M}) \tag{7}$$

The rest of this section will be devoted to the proof of this Theorem.

**Lemma 5.2.** *Let $I$, $I'$ be the defining ideals of $S$, $S'$ in $X$, $X'$. Define $K_{t,S,X}$ as the complex of $\mathcal{O}_X$-modules*

$$\pi_*\left(\mathcal{O}_{X'}/I'^t\right) \xrightarrow{d_0} \pi_*\left(\mathcal{O}_{X'}/I'^{t-1} \otimes_{\mathcal{O}_{X'}} \Omega_{X'/X}\right) \xrightarrow{d_1} \cdots$$
$$\cdots \xrightarrow{d_{q-1}} \pi_*\left(\mathcal{O}_{X'}/I'^{t-q} \otimes_{\mathcal{O}_{X'}} \wedge^q\Omega_{X'/X}\right) \xrightarrow{d_q} \cdots$$

*where $d_q$ is obtained from the exterior differentiation $d : \wedge^q\Omega_{X'/X} \to \wedge^{q+1}\Omega_{X'/X}$.*

*The canonical map $\mathcal{O}_X/I^t \xrightarrow{u} \pi_*\left(\mathcal{O}_{X'}/I'^t\right)$ defines a complex*

$$0 \to \mathcal{O}_X/I^t \xrightarrow{u} K_{t,S,X} \tag{8}$$

*Let $\mathcal{J}$ be a quasicoherent injective $\mathcal{O}_X$-module. Then the complex obtained by applying* $\operatorname{inj\,lim} t\operatorname{Hom}(-, \mathcal{J})$ *to (8) is exact.*

We will not prove this lemma directly. Instead we will treat a special case first.

**Lemma 5.3.** *If $\phi$ is an isomorphism and $S$ is smooth then (8) is exact.*

*Proof.* $\pi_*$ is exact on coherent modules with support in $S$. Hence (8) may be filtered in such a way that the associated graded complexes are of the form

$$0 \to I^{s-1}/I^s \xrightarrow{\bar{u}} \pi_*\left(I'^{s-1}/I'^s\right) \xrightarrow{\bar{d}_0} \pi_*\left(I'^{s-2}/I'^{s-1} \otimes \Omega_{X'/X}\right) \xrightarrow{\bar{d}_1} \cdots \tag{9}$$

Then for (8) to be exact, it is sufficient that (9) is exact for all $s$. Furthermore the hypothesis imply that the sequences (9) are sequences of vector bundles on $\mathcal{O}_S$.

Since $S$ and $S'$ are smooth, we know that $I^{s-1}/I^s = S^{s-1}(I/I^2)$ and $I'^{s-1}/I'^s = S^{s-1}(I'/I'^2)$. With these identifications, the differentiation $\overline{d}_q$ is given by (on an affine open set)

(10)

$$\overline{d}_q(a_1Sa_2\cdots Sa_{s-q}\otimes db_1\wedge\cdots\wedge db_q) = \sum_{i=1}^{s-q} a_1S\cdots S\hat{a}_iS\cdots Sa_{s-q}\otimes da_i\wedge db_1\wedge\cdots\wedge db_q$$

For $s = 1$ the sequence (9) reads as

$$0 \to \mathcal{O}_S \to \pi_*\mathcal{O}_{S'} \to 0$$

which is obviously exact.

For $s = 2$ we obtain

$$0 \to I/I^2 \to \pi_*(I'/I'^2) \to \pi_*(\Omega_{X'/X} \otimes \mathcal{O}_{S'}) \to 0 \tag{11}$$

Exactness of this sequence is obtained from the following diagram :

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \pi_*(\Omega_{X'/X} \otimes \mathcal{O}_{S'}) & & & & \\
 & & & & \uparrow & & & & \\
0 & \to & \pi_*(I'/I'^2) & \to & \pi_*(\Omega_{X'/k} \otimes \mathcal{O}_{S'}) & \to & \pi_*(\Omega_{S'/k}) & \to & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \to & I/I^2 & \to & \Omega_{X/k} \otimes \mathcal{O}_S & \to & \Omega_{S/k} & \to & 0 \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Here the vertical exact sequence is obtained by applying $\pi_*(- \otimes \mathcal{O}_{S'})$ to the standard exact sequence :

$$0 \to \pi^*(\Omega_{X/k}) \to \Omega_{X'/k} \to \Omega_{X'/X} \to 0$$

for smooth maps.

Finally for $s \geq 2$ one deduces from (10) that (9) is obtained from (11) by taking exterior powers. Hence (9) is exact for $s \geq 2$. □

*Proof.* **of Lemma 5.2** Our proof will be by induction on the dimension of $S$. We may clearly reduce to the case where $S$ and $S'$ are reduced. In that case there will be an open subvariety $S'_1$ of $S'$ such that $\phi|S'_1$ is an isomorphism. By making $S'_1$ smaller if necessary we can also assume that $S'_1$ is smooth. Define $S'_2 = S' \setminus S'_1$, $S_1 = \phi(S'_1)$, $S_2 = \phi(S'_2)$, $X_1 = X \setminus S_2$ and $X'_1 = X \setminus S'_2$.

By our induction hypothesis and by lemma 5.3 we may assume that 5.2 has been proved in the situations :

$$\begin{array}{ccc} S_1' & \rightarrow & X_1' \\ \downarrow \phi & & \downarrow \pi \\ S_1 & \rightarrow & X_1 \end{array} \qquad \begin{array}{ccc} S_2' & \rightarrow & X' \\ \downarrow \phi & & \downarrow \pi \\ S_2 & \rightarrow & X \end{array}$$

To complete our induction step, we need another lemma :

**Lemma 5.4.** *Let $X$ be a Noetherian scheme and let $U \subset X$ be an open subset. Let $I$ be the defining ideal of $X \setminus U$. Then for a quasicoherent injective $\mathcal{O}_X$-module $\mathcal{J}$ and a coherent $\mathcal{O}_X$-module $\mathcal{M}$ the following sequence is exact*

$$0 \rightarrow \operatorname{inj\,lim} s \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}/I^s\mathcal{M}, \mathcal{J}) \rightarrow Hom_{\mathcal{O}_X}(\mathcal{M}, \mathcal{J}) \rightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{M}|U, \mathcal{J}|U) \rightarrow 0$$

*where the maps are the obvious ones.*

*Proof.* Well known and easy. □

Now let $I_2$, $I_2'$ be the defining ideals of $S_2$,$S_2'$ in $X$, $X'$. From the lemma we obtain exact sequences

$$0 \rightarrow \operatorname{inj\,lim} t \operatorname{inj\,lim} s \operatorname{Hom}_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{X'}/I'^t \otimes \wedge^q \Omega_{X'/X}) \otimes \mathcal{O}_X/I_2^s, \mathcal{J}) \rightarrow$$
$$\operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{X'}/I'^t \otimes \wedge^q \Omega_{X'/X}), \mathcal{J}) \rightarrow$$
$$\operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_{X_1}}(\pi_*(\mathcal{O}_{X'}/I'^t \otimes \wedge^q \Omega_{X'/X})|X_1, \mathcal{J}|X_1) \rightarrow 0$$

But by a standard argument :

$$\operatorname{inj\,lim} t \operatorname{inj\,lim} s \operatorname{Hom}_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{X'}/I'^t \otimes \wedge^q \Omega_{X'/X}) \otimes \mathcal{O}_X/I_2^s, \mathcal{J})$$
$$= \operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{X'}/I'^t \otimes \wedge^q \Omega_{X'/X}) \otimes \mathcal{O}_X/I_2^t, \mathcal{J})$$
$$= \operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{X'}/(I'^t + \pi^* I_2^t) \otimes \wedge^q \Omega_{X'/X}), \mathcal{J})$$

But the chains of ideals $(I'^t + \pi^* I_2^t)_t$ and $(I_2'^t)_t$ are cofinal in each other.

We obtain exact sequences :

$$0 \rightarrow \operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{X'}/I_2'^t \otimes \wedge^q \Omega_{X'/X}), \mathcal{J}) \rightarrow$$
$$\operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{X'}/I'^t \otimes \wedge^q \Omega_{X'/X}), \mathcal{J})$$
$$\rightarrow \operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_{X_1}}(\pi_*(\mathcal{O}_{X'}/I'^t \otimes \wedge^q \Omega_{X'/X})|X_1, \mathcal{J}|X_1) \rightarrow 0$$

In a similar but easier way one obtains from 5.4 that

$$0 \rightarrow \operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X/I_2^t, \mathcal{J}) \rightarrow$$
$$\operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X/I^t, \mathcal{J}) \rightarrow$$
$$\operatorname{inj\,lim} t \operatorname{Hom}_{\mathcal{O}_{X_1}}(\mathcal{O}_X/I^t|X_1, \mathcal{J}|X_1) \rightarrow 0$$

is exact.

We can combine these sequences into a diagram :

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \to & \operatorname{inj\,lim} t\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X/I_2^t,\mathcal{J}) & \to & \operatorname{inj\,lim} t\mathrm{Hom}_{\mathcal{O}_X}(K^{\cdot}_{t,S_2,X},\mathcal{J}) \\
 & & \downarrow & & \downarrow \\
0 & \to & \operatorname{inj\,lim} t\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X/I^t,\mathcal{J}) & \to & \operatorname{inj\,lim} t\mathrm{Hom}_{\mathcal{O}_X}(K^{\cdot}_{t,S,X},\mathcal{J}) \\
 & & \downarrow & & \downarrow \\
0 & \to & \operatorname{inj\,lim} t\mathrm{Hom}_{\mathcal{O}_{X_1}}(\mathcal{O}_X/I^t|X_1,\mathcal{J}|X_1) & \to & \operatorname{inj\,lim} t\mathrm{Hom}_{\mathcal{O}_{X_1}}(K^{\cdot}_{t,S_1,X_1},\mathcal{J}|X_1) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

It follows now from our induction hypothesis that the middle complex is exact. □

*Proof.* **of Theorem 5.1**

We start with an injective resolution $0 \to \mathcal{M} \to \mathcal{J}^{\cdot}$. We then obtain a double complex

$$\operatorname{inj\,lim} t\mathrm{Hom}(K^{\cdot}_{t,S,X},\mathcal{J}^{\cdot}) \tag{12}$$

which we think of as lying in the first quadrant such that the maps obtained from $K_{t,S,X}$ are horizontal, and such that the lower lefthand corner is $\operatorname{inj\,lim} t\mathrm{Hom}(K^{\dim(X'/X)}_{t,S,X},\mathcal{J}^0)$.

To compute the homology of (12) we use the first filtration. By lemma 5.2 we obtain the complex

$$\Gamma_S(X,\mathcal{J}^{\cdot}) \tag{13}$$

at horizontal position $\dim(X'/X)$ and zero's everywhere else. The homology of (13) is clearly $H^q_S(X,\mathcal{M})$ at position $(\dim(X'/X),q)$. Hence (12) has homology $H^q_S(X,\mathcal{M})$ at position $q+\dim(X'/X)$.

To use the second filtration we have to compute the homology of the complexes

$$\mathrm{Hom}_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{X'}/I'^{t-p'}\otimes_{\mathcal{O}_{X'}}\wedge^{p'}\Omega_{X'/X}),\mathcal{J}^{\cdot})$$

where $p'=\dim(X'/X)-p$.

We obtain

$$\mathrm{Ext}^q_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{X'}/I'^{t-p'}\otimes_{\mathcal{O}_{X'}}\wedge^{p'}\Omega_{X'/X}),\mathcal{M})$$

which is by duality [6, Thm. III.11.1]:

$$\mathrm{Ext}^{q+\dim(X'/X)}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}/I'^{t-p'}\otimes_{\mathcal{O}_{X'}}\wedge^{p'}\Omega_{X'/X},\pi^*\mathcal{M}\otimes\omega_{X'/X}) \tag{14}$$

where we have used the fact that $R\pi_* = 0$ on modules with support in $S'$.

Simplifying (14) further we obtain

$$\begin{aligned}
&\mathrm{Ext}^{q+\dim(X'/X)}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}/I'^{t-p'} \otimes_{\mathcal{O}_{X'}} \wedge^{p'}\Omega_{X'/X}, \pi^*\mathcal{M}\otimes\omega_{X'/X})\\
&= \mathrm{Ext}^{q+\dim(X'/X)}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}/I'^{t-p'}, \pi^*\mathcal{M}\otimes\omega_{X'/X}\otimes(\wedge^{p'}\Omega_{X'/X})^*)\\
&= \mathrm{Ext}^{q+\dim(X'/X)}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}/I'^{t-p'}, \pi^*\mathcal{M}\otimes\wedge^{\dim(X'/X)-p'}\Omega_{X'/X})
\end{aligned}$$

Hence after taking homology for the second filtration in (12) we obtain a diagram with

$$H^{q+\dim(X'/X)}_{S'}(X', \pi^*\mathcal{M}\otimes_{\mathcal{O}_{X'}}\wedge^p\Omega_{X'/X})$$

at position $(p,q)$. After reindexing we obtain (7). $\square$

## 6. Constructibility.

In this section we will use Theorem 5.1 to get some results on cohomology with support in the unstable locus. Roughly speaking, we will decompose the unstable locus as a union of a closed and a locally closed subvariety, which can be handled by 5.1. Then we use induction. It would be natural to try to use the well known stratification into smooth subvarieties, due to Hesselink [7], Kirwan [9] and others. Unfortunately this stratification turns out to be too fine for our purposes. The decomposition we must use is much coarser and the parts are not necessarily smooth. What is worse however is that it does not always work ! This leads us to a concept we call constructibility and which is introduced below.

As usual $G$ will be a reductive algebraic group with a Borel subgroup $B \subset G$ containing a maximal torus $T \subset G$. $W$ will be a $G$-representation and $R = k[W]$. We define furthermore $X = \mathrm{Spec} R$, $X/G = \mathrm{Spec} R^G$, $d = \dim W$, $h = \dim R^G$. The roots of $B$ will be the negative roots and $\Phi^+$ will denote the set of positive roots.

### 6.1. **Reduction pairs and constructibility.**

In the sequel, a pair $(P, Y)$ will consist of a parabolic subgroup $P \subset G$ containing $B$, and a linear subspace $Y$ of $X$ which is preserved by $B$.

**Definition 6.1.1.** A reduction pair for $(P, Y)$ is a pair $(P_1, Y_1)$ such that

1. $P_1 \subset P$, $Y_1 \subset Y$ and the inclusions are strict.
2. $(P \setminus P_1)Y \cap Y \subset P_1 Y_1$

We first have to introduce some more notations : if $P$ is a parabolic subgroup of $G$ then $f : G \times^P X \to G/P$ will be the natural projection map $\overline{(g,x)} \mapsto \overline{g}$.

If $P_2 \supset P_1$ are parabolic subgroups of $G$ then for $l \geq 0$ we will denote with $\Omega^l_{[P_2/P_1]}$ the $\mathcal{O}_{G/P_1}$-module $\wedge^l \Omega_{(G/P_1)/(G/P_2)}$.

From the fact that there is a commutative diagram

$$\begin{array}{ccc} G \times^P X & \to & G/P \times X \\ \downarrow & & \downarrow \\ G/P & = & G/P \end{array}$$

for any subgroup $P$ of $G$ we deduce that $\wedge^l \Omega_{(G\times^{P_1}X)/(G\times^{P_2}X)} = f^*\Omega^l_{[P_2/P_1]}$.

The quotient map $G/P_1 \to G/P_2$ will be denoted by $\pi^{P_1}_{P_2}$. The same notation is used for the analogous map $G \times^{P_1} X \to G \times^{P_2} X$.

$l(P_2/P_1)$ will be the biggest $u$ such that there is a chain $P_2 = P^{(u)} \supset P^{(u-1)} \supset \cdots \supset P^{(0)} = P_1$ of parabolics, such that all inclusions are strict. Clearly $l(G/B)$ is the rank of the semisimple part of $G$.

Finally if $P_u \supset P_{u-1} \supset \cdots \supset P_0$ is a chain of parabolic subgroups of $G$ then we define $\Omega^{l_u \cdots l_1}_{[P_u/P_{u-1}\cdots/P_0]}$ for natural numbers $(l_i)_i$ as

$$\Omega^{l_1}_{[P_1/P_0]} \otimes \pi^{P_0*}_{P_1}\Omega^{l_2}_{[P_2/P_1]} \otimes \cdots \otimes \pi^{P_0*}_{P_{u-1}}\Omega^{l_u}_{[P_u/P_{u-1}]}$$

**Lemma 6.1.2.** *Assume that $(P,Y)$ is a pair and that $(P_1,Y_1)$ is a reduction pair. Let $\mathcal{M}$ be a $G$-equivariant, quasicoherent $\mathcal{O}_{G\times^P X}$-module. Then every $G$-representation that occurs in $H^i_{G\times^P PY}(G \times^P X, \mathcal{M})$, occurs in one of the following $G$-modules.*

(1) $H^i_{G\times^P PY_1}(G \times^P X, \mathcal{M})$
(2) $H^{i_1+2\dim(P/P_1)}_{G\times^{P_1}P_1Y}(G \times^{P_1} X, \pi^{P_1*}_P\mathcal{M} \otimes f^*\Omega^{i_2}_{[P/P_1]})$ *where* $i_1 + i_2 = i$
(3) $H^{i_1+1+2\dim(P/P_1)}_{G\times^{P_1}P_1Y_1}(G \times^{P_1} X, \pi^{P_1*}_P\mathcal{M} \otimes f^*\Omega^{i_2}_{[P/P_1]})$ *where* $i_1 + i_2 = i$

*Proof.* By the standard long exact sequence for cohomology with support, any representation occurring in $H^i_{G\times^P PY}(G \times^P X, \mathcal{M})$ must also occur in $H^i_{G\times^P PY_1}(G \times^P X, \mathcal{M})$ or in

$$H^i_{G\times^P (PY\setminus PY_1)}(G \times^P (X \setminus PY_1), \mathcal{M}) \quad (15)$$

By 1. we only have to concern ourselves with the latter case. We first prove a sublemma.

**Lemma 6.1.3.** *The commutative diagram*

$$\begin{array}{ccc} G \times^{P_1} P_1Y & \to & G \times^{P_1} X \\ \downarrow & & \downarrow \\ G \times^P PY & \to & G \times^P X \end{array}$$

*restricts to a diagram*

$$
(16)\qquad \begin{array}{ccc} G\times^{P_1}(P_1Y\setminus P_1Y_1) & \xrightarrow{\alpha} & G\times^{P_1}(X\setminus PY_1)\\ \downarrow{\scriptstyle\beta} & & \downarrow\\ G\times^{P}(PY\setminus PY_1) & \rightarrow & G\times^{P}(X\setminus PY_1)\end{array}
$$

*where $\alpha$ is a closed immersion and $\beta$ is a bijection.*

*Proof.* This can be deduced from 2. in definition 6.1.1. We first show that $Y\setminus PY_1 = Y\setminus P_1Y_1$ which is equivalent with $Y\cap PY_1 = Y\cap P_1Y_1$.

$$
\begin{aligned} Y\cap PY_1 &= (Y\cap P_1Y_1)\cup(Y\cap(P\setminus P_1)Y)\\ &= Y\cap P_1Y_1\end{aligned}
$$

From this it follows that $\alpha$ and $\beta$ are defined. For $\alpha$ we have to show that $P_1Y\setminus P_1Y_1$ is a closed subset of $X\setminus PY_1$. But $P_1Y\setminus P_1Y_1 = P_1(Y\setminus P_1Y_1) = P_1(Y\setminus PY_1) = P_1Y\setminus PY_1 \subset X\setminus PY_1$ and the last inclusion is closed.

Similarly for $\beta$ we first have to show that $P(P_1Y\setminus P_1Y_1)\subset PY\setminus PY_1$. Again $P(P_1Y\setminus P_1Y_1) = P(Y\setminus P_1Y_1) = P(Y\setminus PY_1) = PY\setminus PY_1$. This also shows that $\beta$ is surjective.

To show that $\beta$ is a bijection let $y\in Y\setminus PY_1$. Any other element in $G\times^P(PY\setminus PY_1)$ is in the $G$-orbit of such an element. A quick check then shows ($\cong$ means : "there is a bijection")

$$
\beta^{-1}(y)\cong\{(p,y')\in P\times(P_1Y\setminus P_1Y_1)|py'=y\}/P_1
$$

Hence

$$
\begin{aligned} \beta^{-1}(y) &\cong \{p\in P\mid p^{-1}y\in P_1Y\setminus P_1Y_1\}/P_1\\ &\cong \{p\in P\mid p^{-1}y\in P_1Y\setminus PY_1\}/P_1 \text{ (as above)}\\ &\cong \{p\in P\mid y\in pP_1Y\setminus PY_1\}/P_1\\ &\cong \{p\in P\mid y\in pP_1Y\}/P_1 \text{ (since } y\notin PY_1 \text{ by hyp.)}\\ &\cong \text{ singleton (using 2. in 6.1.1)}\end{aligned}
$$

□

*Remark* 6.1.4. One can actually prove that the existence of diagram (16), together with the fact that $\beta$ is an isomorphism, is equivalent with condition 2. in 6.1.1.

Now we continue with the proof of lemma 6.1.2.

By 5.1 and diagram (16) every representation that occurs in (15) must occur in one of the representations :

$$
(17)\qquad H^{i_1+2\dim(P/P_1)}_{G\times^{P_1}(P_1Y\setminus P_1Y_1)}(G\times^{P_1}(X\setminus PY_1),\pi_P^{P_1 *}\mathcal{M}\otimes f^*\Omega^{i_2}_{[P/P_1]})
$$

where $i_1 + i_2 = i$.

But $G \times^{P_1} (X \setminus P_1Y_1)$ is an open subset of $G \times^{P_1} X$ containing $G \times^{P_1} (X \setminus PY_1)$ and $G \times^{P_1} (P_1Y \setminus P_1Y_1)$ is still a closed subset of $G \times^{P_1} (X \setminus P_1Y_1)$. Therefore (17) is equal to (excision)

$$H^{i_1+2\dim(P/P_1)}_{G\times^{P_1}(P_1Y\setminus P_1Y_1)}(G \times^{P_1} (X \setminus P_1Y_1), \pi_P^{P_1*}\mathcal{M} \otimes f^*\Omega^{i_2}_{[P/P_1]}) \tag{18}$$

Invoking again the long exact cohomology sequence, yields that any representation occurring in (18) must occur in

$$H^{i_1+2\dim(P/P_1)}_{G\times^{P_1}P_1Y}(G \times^{P_1} X, \pi_P^{P_1*}\mathcal{M} \otimes f^*\Omega^{i_2}_{[P/P_1]})$$

or in

$$H^{i_1+1+2\dim(P/P_1)}_{G\times^{P_1}P_1Y_1}(G \times^{P_1} X, \pi_P^{P_1*}\mathcal{M} \otimes f^*\Omega^{i_2}_{[P/P_1]})$$

□

**Definition 6.1.5.** A pair $(P, Y)$ is constructible if and only if one of the following holds

- $PY = Y$ and $Y = X_\lambda$ for some $\lambda \in Y(T)$ (including 0) belonging to the Weyl chamber determined by $B$.
- There exists a reduction pair $(P_1, Y_1)$ for $(P, Y)$ such that $(P_1, Y_1)$, $(P_1, Y)$ and $(P, Y_1)$ are constructible.

**Proposition 6.1.6.** *Assume that the pair $(G, Y)$ is constructible. Then any $G$-representation occurring in*

$$H^i_{GY}(X, \mathcal{O}_X)$$

*occurs in some*

$$H^j_{G\times^B Y'}(G \times^B X, f^*\pi_{P'}^{B*}\Omega^{i_u\cdots i_1}_{[P_u/P_{u-1}/\cdots/P_0]})$$

*where*

- *$(P', Y')$ is a pair such that $P' \subset G$, $Y' \subset Y$, $PY' = Y'$, $Y' = X_\lambda$ and $\lambda$ is in the Weyl chamber determined by $B$.*
- *$P_u \supset P_{u-1} \supset \cdots \supset P_0$ is a chain (with strict inclusions) such that $G = P_u$, $P' = P_0$.*
- *$i \le j + i_1 + \cdots + i_u \le i + u + 2\dim(G/P')$*

*Proof.* This follows by induction from lemma 6.1.2 and the observation that $H^{\cdot}_{G\times^{P'}Y'}(G\times^{P'} X, f^*\Omega^{i_u\cdots i_1}_{[P_u/P_{u-1}/\cdots/P_0]}) = H^{\cdot}_{G\times^B Y'}(G\times^B X, f^*\pi_{P'}^{B*}\Omega^{i_u\cdots i_1}_{[P_u/P_{u-1}/\cdots/P_0]})$. □

**Proposition 6.1.7.** *Let $(G, Y)$ be a constructible pair and let $\chi \in X(T)$ be a dominant character (with respect to $\Phi^+$). Assume that $\chi - w(\sum_{\rho\in S} \rho)$ is strongly critical for $(T, W)$, for all $w \in \mathcal{W}_G$, $S \subset -\Phi^+$. Then there will be no $G$-representation with highest weight $\chi$ in $H^i_{GY}(X, \mathcal{O}_X)$ for $0 \le i \le d - l(G/B) - 2\dim(G/B) - 1$.*

*Proof.* By Prop. 6.1.6 it is sufficient to prove the same statement for

$$H^j_{G\times^B Y'}(G\times^B X, f^*\pi^{B*}_{P'}\Omega^{i_u\cdots i_1}_{[P_u/P_{u-1}/\cdots/P_0]}) \tag{19}$$

where $0\le j+i_u+\cdots+i_1\le d-l(G/B)-2\dim(G/B)-1+u+2\dim(G/P')\le d-1$. This is obviously true if $Y'=\{0\}$. Hence we assume that $Y'\neq\{0\}$.

(19) is equal to

$$H^j_{G\times^B Y'}(G\times^B X, \mathcal{O}_{G\times^B X}\otimes_{\mathcal{O}_{G/B}}\pi^{B*}_{P'}\Omega^{i_u\ldots i_1}_{[P_u/\cdots/P_0]})$$

If we take the fiber of $\pi^{B*}_{P'}\Omega^{i_u\ldots i_1}_{[P_u/\cdots/P_0]}$ over $[B]$ then we obtain

$$\wedge^{i_u}(\mathbf{p}_u/\mathbf{p}_{u-1})^*\otimes\cdots\otimes\wedge^{i_1}(\mathbf{p}_1/\mathbf{p}_0)^* \tag{20}$$

where $\mathbf{p}_j$ is the Lie algebra of $P_j$.

(20) is a $B$-representation which has a filtration whose associated quotient representations are $T$-characters of the form

$$\chi_1^{(S)}=\sum_{\rho\in S}\rho \text{ where } S\subset-\Phi^+$$

Using lemma 6.2.1 of the next section it is sufficient that $\chi-w\chi_1^{(S)}$ is strongly critical for $(T,W)$ for all $w\in\mathcal{W}_G$ and for all $S\subset-\Phi^+$. But this was exactly the hypothesis. □

It is well known that every fiber of $X\to X/G$ contains a unique closed orbit. A point $x\in X$ is called stable if for all $\lambda\in Y(G)$ neither $\lim_{t\to0}\lambda(t)x$ nor $\lim_{t\to\infty}\lambda(t)x$ exists. Stable points have finite stabilizer and closed $G$-orbit. They form an invariant open subset of $X$.

Hence one deduces that $\dim X=\dim(X/G)+\dim G$ if there is at least one stable point in $X$.

Let us also recall the following theorem :

**Theorem 6.1.8.** [15] *Assume that a semisimple group $G$ acts on an affine variety $X$ with factorial coordinate ring such that the generic stabiliser is finite. Then $X$ has a stable point.*

To simplify the notation a bit we will say that $X^u$ is constructible if it is of the form $GX_\lambda$, where $(G,X_\lambda)$ is a constructible pair.

**Theorem 6.1.9.** *Let $G$ be semisimple and assume that $X^u$ is constructible. Assume furthermore that $X$ has a $G$-stable point. Then Conj. 3.4' is true.*

*Proof.* Let $\chi$ be a character of $G$ and let $\chi_1\in X(T)$ be the highest weight of its corresponding $G$-representation. The hypothesis for Conj. 3.4' imply that $\chi_1-w(\sum_{\rho\in S}\rho)$ will be strongly critical for

$(T, W)$ for all $w \in \mathcal{W}_G$, $S \subset -\Phi^+$. Hence by Prop. 6.1.7 there will be no representation with character $\chi$ in $H^i_{X^u}(X, \mathcal{O}_X)$ for $0 \le i \le d - l(G/B) - 2\dim(G/B) - 1$.

However since $G$ is assumed to be semisimple and $X$ has a stable point, $l(G/B) + 2\dim(G/B) = \dim G$ and $d - \dim G - 1 = h - 1$. Hence the conditions for Cor. 4.2 are satisfied, and therefore $R^G_\chi$ is Cohen Macaulay. □

*Remark* 6.1.10. Note that in Prop. 6.1.7 one actually proves more than in Th. 6.1.9. However I have no example where this makes any difference.

**Theorem 6.1.11.** *Conj. 3.4' is true in the case* $G = SL(V)$*,* $\dim V = 2$*.*

*Proof.* We may assume that $W$ does not contain trivial representations.

Assume first that $W = V$ or $W = S^2V$. Then $k[W]^G$ is a PID. Hence $R^G_\chi$ is a torsion free module over a PID, and therefore Cohen-Macaulay. This means that Conj. 3.4' is vacuous and hence true.

Assume now that $W \neq V$ and $W \neq S^2V$. Then $X$ has a stable point by 6.1.8. From the fact that $X(T) = \mathbb{Z}$ one deduces that $X^u = GX_\lambda$ where $\lambda(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$. From definition 6.1.1 or else from (26) below one deduces that $(B, \{0\})$ is a reduction pair for $(G, X_\lambda)$. Since $(B, X_\lambda)$, $(G, \{0\})$ and $(B, \{0\})$ are constructible by the first condition of definition 6.1.5 we deduce that $(G, X_\lambda)$ is constructible. □

6.2. **Some computations.** The following lemma was used in the proof of Prop. 6.1.7. This subsection will be devoted to its proof.

**Lemma 6.2.1.** *Assume that* $\lambda \in Y(T)$ *belongs to the Weyl chamber determined by* $B$*. Let* $\chi$*,* $\chi_1 \in X(T)$ *where* $\chi$ *is dominant and* $\chi - w\chi_1$ *is strongly critical for* $(T, W)$*, for all* $w \in \mathcal{W}_G$*. Assume furthermore that* $X$ *has a stable point. Then no* $G$*-representation with highest weight* $\chi$ *occurs in*

$$H^{\cdot}_{G\times^B X_\lambda}(G \times^B X, \mathcal{O}_{G\times^B X} \otimes_{\mathcal{O}_{G/B}} \tilde{\chi}_1) \tag{21}$$

*Proof.* Let $Y = X_\lambda$. There is a spectral sequence :

$$E_2^{pq} : H^p(\mathcal{H}^q_{G\times^B Y}(G \times^B X, \mathcal{O}_{G\times^B X} \otimes_{\mathcal{O}_{G/B}} \tilde{\chi}_1))$$

$$\Rightarrow H^{p+q}_{G\times^B Y}(G \times^B X, \mathcal{O}_{G\times^B X}, \mathcal{O}_{G\times^B X} \otimes_{\mathcal{O}_{G/B}} \tilde{\chi}_1) \tag{22}$$

Furthermore, it is easy to see that

$$\mathcal{H}^q_{G\times^B Y}(G \times^B X, \mathcal{O}_{G\times^B X} \otimes_{\mathcal{O}_{G/B}} \tilde{\chi}_1) = \mathcal{H}^q_{G\times^B Y}(G \times^B X, (\mathcal{O}_X \otimes_k \chi_1)\tilde{)}$$

$$= (H^q_Y(X, \mathcal{O}_X) \otimes_k \chi_1)\tilde{} \tag{23}$$

as $\mathcal{O}_{G/B}$-modules.

Let $J$ be the defining ideal of $Y$. $J$ is generated by a subspace $W'$ of $W$. Define $W'' = W/W'$, $d' = \dim W'$. We need the following result :

**Lemma 6.2.2.**

- *$H^i_J(R) = 0$ if $i \neq d'$.*
- *$H^{d'}_J(R)$ is, as a $T$-representation, isomorphic to $(\wedge^{d'} W')^* \otimes \oplus_{n=0}^{\infty} S^n(W'^* \oplus W'')$*

*Proof.* The first statement is clear since $J$ is generated by a system of parameters.

For the second statement we use the fact that $H^{d'}_J(R) = \operatorname{inj\,lim} t\operatorname{Ext}_R(R/J^t, R)$. We first compute $\operatorname{Ext}_R(J^t/J^{t+1}, R) \cong (S^t W')^* \otimes \operatorname{Ext}^i_R(R/J, R)$. Again $\operatorname{Ext}^i(R/J, R) = 0$ if $i \neq d'$. On the other hand, using the Koszul resolution for $R/J$, one easily computes that $\operatorname{Ext}^{d'}_R(R/J, R) \cong (\wedge^{d'} W')^* \otimes R/J$. Hence as $T$-module : $H^i_J(R) = \oplus_t \oplus_{t'} (\wedge^{d'} W')^* \otimes (S^t W')^* \otimes S^{t'} W'' = (\wedge^{d'} W')^* \otimes \oplus_{t \geq 0} S^t(W'^* \oplus W'')$. □

Now assume that $W = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_d$ as $T$-representation, where $\alpha_i \in X(T)$. Define $I = \{i \in \{1, \ldots, d\} \mid \langle \lambda, \alpha_i \rangle \geq 0\}$. Then by construction, the weights of $W'$ and $W''$ are resp. $(\alpha_i)_{i \in I}$ and $(\alpha_i)_{i \notin I}$. Therefore

$$H^{d'}_Y(X, \mathcal{O}_X) \otimes \chi_1$$

has a filtration (as $B$-representation) whose associated graded quotients $\chi'$ are by lemma 6.2.2 of the form

$$-\sum_{i \in I} \alpha_i - \sum_{i \in I} a_i \alpha_i + \sum_{i \notin I} b_i \alpha_i + \chi_1 \tag{24}$$

where $a_i$, $b_i \in \mathbb{N}$.

Hence they have the property that

$$\langle \lambda, \chi' - \chi_1 \rangle = \langle \lambda, -\sum_{i \in I} \alpha_i \rangle - \sum_{i \in I} a_i \langle \lambda, \alpha_i \rangle + \sum_{i \notin I} b_i \langle \lambda, \alpha_i \rangle \leq -\sum_{i \in I} \langle \lambda, \alpha_i \rangle \tag{25}$$

Let $\overline{\rho} \in X(T)_{\mathbb{Q}}$ be half the sum of the positive roots.

Now assume that $\chi$ does occur in (21). Then it must also occur somewhere in the $E_2$-term of the spectral sequence (22). Hence, by Bott's theorem [3] and by (23) it must be of the form $w(\chi' + \overline{\rho}) - \overline{\rho}$ where $w \in \mathcal{W}_G$, and $\chi'$ is of the form (24).

From (25) we deduce

$$\begin{aligned} -\sum_{i\in I}\langle\lambda,\alpha_i\rangle &\geq \langle\lambda, w^{-1}(\chi+\overline{\rho})-\overline{\rho}-\chi_1\rangle \\ &= \langle\lambda, w^{-1}(\chi-w\chi_1)+w^{-1}\overline{\rho}-\overline{\rho}\rangle \\ &= \langle\lambda, w^{-1}(\chi-w\chi_1)\rangle+\langle\lambda, w^{-1}\overline{\rho}-\overline{\rho}\rangle \end{aligned}$$

However by 6.2.3 below and by the hypothesis, $w^{-1}(\chi-w\chi_1)$ is strongly critical for $(T,W)$. Hence we will deduce that (6.2.4) $\langle\lambda, w^{-1}(\chi-w\chi_1)\rangle > -\sum_{i\in I}\langle\lambda,\alpha_i\rangle$.

Since furthermore $\langle\lambda, w^{-1}\overline{\rho}-\overline{\rho}\rangle\geq 0$ (6.2.5), we obtain a contradiction. □

Now we will fill in the few missing steps in lemma 6.2.2.

**Lemma 6.2.3.** *If $\chi$ is strongly critical for $(T,W)$ and $w\in\mathcal{W}_G$ then $w(\chi)$ is also strongly critical for $(T,W)$.*

*Proof.* True, because $\mathcal{W}_G$ permutes the weights of $W$. □

**Lemma 6.2.4.** *Assume that $\chi$ is strongly critical for $(T,W)$ and $\lambda\in Y(T)$. Assume furthermore that $X$ has a stable point. Define $I=\{i\in\{1,\dots,d\}\mid\langle\lambda,\alpha_i\rangle\geq 0\}$. Then $\langle\lambda,\chi\rangle > -\sum_{i\in I}\langle\lambda,\alpha_i\rangle$.*

*Proof.* The fact that $X$ has a stable point implies that there exists an $i$ such that $\langle\lambda,\alpha_i\rangle>0$. By definition $\chi=\sum_{i=1}^d a_i\alpha_i$, $-1<a_i\leq 0$. Hence

$$\begin{aligned} \langle\lambda,\chi\rangle &= \sum_{i\in I}a_i\langle\lambda,\alpha_i\rangle+\sum_{i\notin I}a_i\langle\lambda,\alpha_i\rangle \\ &> -\sum_{i\in I}\langle\lambda,\alpha_i\rangle \end{aligned}$$

□

**Lemma 6.2.5.** *Let $\lambda\in Y(T)$ belong to the Weyl chamber determined by $B$. Let $\overline{\rho}$ be as in the proof of lemma 6.2.1. Then $\langle\lambda, w\overline{\rho}-\overline{\rho}\rangle\geq 0$ for all $w\in\mathcal{W}_G$.*

*Proof.* Since $\lambda$ belongs to the Weyl chamber determined by $B$, one deduces easily that $\langle\lambda,\rho\rangle\leq 0$ for all $\rho\in\Phi^+$. On the other hand the definition of $\overline{\rho}$ immediately implies that $w\overline{\rho}-\overline{\rho}$ is a sum of negative roots. □

6.3. **A combinatorial criterion for constructibility.** To verify whether a pair $(P_1,Y_1)$ is a reduction pair for some other pair $(P,Y)$ we need some way of checking condition 2. in 6.1.1. A simple criterion that can be checked on the weights of $W$ is given below.

**Proposition 6.3.1.** *Assume that* $(P_1, Y_1)$, $(P, Y)$ *are pairs such that* $P_1 \subset P$, $Y_1 \subset Y$ *and the inclusions are strict. If*

$$\forall w \in \mathcal{W}_P \setminus \mathcal{W}_{P_1} : wY \cap Y \subset \bigcup_{w' \in \mathcal{W}_{P_1}} w'Y_1 \tag{26}$$

*then* $(P_1, Y_1)$ *is a reduction pair for* $(P, Y)$.

*Proof.* Assume that (26) holds. By the Bruhat decomposition

$$P_1 = B\mathcal{W}_{P_1}B$$

and

$$P \setminus P_1 = B(\mathcal{W}_P \setminus \mathcal{W}_{P_1})B$$

Hence

$$\begin{aligned} (P \setminus P_1)Y \cap Y &= B(\mathcal{W}_P \setminus \mathcal{W}_{P_1})Y \cap Y \\ &= B((\mathcal{W}_P \setminus \mathcal{W}_{P_1})Y \cap Y) \\ &\subset B\mathcal{W}_{P_1}Y_1 \\ &= P_1Y_1 \end{aligned}$$

□

## 7. The case of matrix concomittants.

In this section we will verify the major hypothesis of Th. 6.1.9 for $\mathbb{T}_{m,n}$, namely that $X^u$ is constructible. Using (26) this can be done combinatorially. As a consequence we obtain that $\mathbb{T}_{m,n}$ is Cohen-Macaulay in general (Th. 7.3.6).

We define $G = SL(V)$ where $\dim V = n$, $W = \mathrm{End}(V)^{m*}$ and $X = \mathrm{Spec} k[W]$. $T \subset G$ will be a maximal torus. We will choose a basis in $V$ such that the action of $T$ on $V$ is diagonal, i.e. of the form $\mathrm{diag}(z_1, \dots, z_n)$ where $z_i \in k$ and $z_1 \cdots z_n = 1$.

### 7.1. Ordered partitions.

If $n$ is an integer then an ordered partition $\nu$ of $n$ will be a tuple $(\nu_1, \nu_2, \dots, \nu_u)$ such that $\sum_{i=1}^u \nu_i = n$, $\nu_i \in \mathbb{N}_0$. If $\nu$ is an ordered partition of some unspecified number then that number will be denoted by $\sum \nu_i$. We will also use the empty tuple () as the unique ordered partition of 0.

If $(\nu^{(i)})_{i=1,\dots,v}$ are ordered partitions where $\nu^{(i)} = (\nu_1^{(i)}, \dots, \nu_{u_i}^{(i)})$ then $(\nu^{(1)}, \dots, \nu^{(v)})$ is the ordered partition $(\nu_1^{(1)}, \dots, \nu_{u_1}^{(1)}, \nu_1^{(2)}, \dots, \nu_{u_v}^{(v)})$.

If $\eta$, $\nu$ are two ordered partitions then we say that $\eta$ is a refinement of $\nu$ (notation : $\eta \lhd \nu$) if $\eta = (\eta^{(1)}, \dots, \eta^{(v)})$ where the $\eta^{(i)}$ are ordered partitions and $\nu = (\sum \eta^{(1)}, \dots, \sum \eta^{(v)})$.

The ordered partitions of $n$, ordered by $\lhd$, form a partially ordered set with minimal element $\eta_{\min} = (1, \dots, 1)$ and maximal element $\eta_{\max} = (n)$.

Let $B$ be the Borel subgroup of $G$ consisting of the upper triangular matrices. Then any $\lambda \in Y(T)$ belonging to the Weyl chamber determined by $B$ will be of the form

$$z \to \operatorname{diag}(\underbrace{z^{i_1}, \dots, z^{i_1}}_{\eta_1 \text{ times}}, \underbrace{z^{i_2}, \dots, z^{i_2}}_{\eta_2 \text{ times}}, \dots, \underbrace{z^{i_u}, \dots, z^{i_u}}_{\eta_u \text{ times}}) \tag{27}$$

where $i_1 > i_2 > \cdots > i_u$ and $\eta_1 i_1 + \eta_2 i_2 + \cdots + \eta_u i_u = 0$. With a slight abuse of notation we will denote this one-parameter subgroup by $\lambda_\eta$, where $\eta$ is the ordered partition of $n$ given by $(\eta_1, \dots, \eta_u)$. This notation is justified in this context by the fact that if $\lambda_1, \lambda_2 \in Y(T)$ are of the form (27) with the same numbers $\eta_1, \dots, \eta_u$ then $G(\lambda_1) = G(\lambda_2)$ and $X_{\lambda_1} = X_{\lambda_2}$.

Clearly $B = G(\lambda_{\eta_{\min}})$ and $G = G(\lambda_{\eta_{\max}})$.

**Lemma 7.1.1.** *If $\eta \lhd \nu$ then*

- $G(\lambda_\eta) \subset G(\lambda_\nu)$
- $X_{\lambda_\eta} \supset X_{\lambda_\nu}$
- $G(\lambda_\eta) X_{\lambda_\nu} = X_{\lambda_\nu}$

*Proof.* This follows by inspection. □

**Lemma 7.1.2.** $X^u = G X_{\lambda_{\eta_{\min}}}$

*Proof.* This follows by lemma 7.1.1 and by lemma 4.3. □

Now define $Q = \{1, \dots, n\}$. If $\nu$ is an ordered partition of $n$ then $Q_\nu$ will be the partition of $Q$ given by $\{\{1, \dots, \nu_1\}, \{\nu_1 + 1, \dots, \nu_1 + \nu_2\}, \dots, \{\nu_1 + \cdots + \nu_{u-1}, \dots, n\}\}$. The elements of $Q_\nu$ will be indexed as $Q_{\nu,i}$ where $Q_{\nu,i} = \{\nu_1 + \cdots + \nu_{i-1}, \dots, \nu_1 + \cdots + \nu_i\}$.

If $T$ is an arbitrary set then $S_T$ will be the permutation group of $T$. If $Q_\nu$ is a partition of $Q$ then $S_{Q_\nu} = \prod_{T \in Q_\nu} S_T$. $Q_\nu^\circ$ will be the set $\bigcup_{j>i} Q_{\nu,i} \times Q_{\nu,j} \subset Q \times Q$.

It is easily verified that if $\nu$ is an ordered partition then

$$\mathcal{W}_{P(\lambda_\nu)} = S_{Q_\nu} \tag{28}$$

and the weights of $X_{\lambda_\nu}$ (considered as a subspace of $W^*$) are

$$\{(z_i z_j^{-1})_{(i,j) \in Q_\nu^\circ}\} \tag{29}$$

**Lemma 7.1.3.** *Let $(\eta_1, \nu_1)$, $(\eta, \nu)$ be pairs of ordered partitions. Assume that $\eta_1 \lhd \eta$, $\nu \lhd \nu_1$ and $\eta_1 \neq \eta$, $\nu_1 \neq \nu$. Suppose that*

$$\forall w \in S_{Q_\eta} \setminus S_{Q_{\eta_1}} : \exists w' \in S_{Q_{\eta_1}} : w Q_\nu^\circ \cap Q_\nu^\circ \subset w' Q_{\nu_1}^\circ \tag{30}$$

*Then* $(G(\lambda_{\eta_1}), X_{\lambda_{\nu_1}})$ *is a reduction pair for* $(G(\lambda_\eta), X_{\lambda_\nu})$.

*Proof.* This is just a translation of (26) to the present situation using (28) and (29). □

7.2. **Settheoretic computations.** In this subsection we will verify (30) for certain special pairs of partitions.

For the sequel let $\eta = (\eta^{(1)}, \eta^{(2)}, x)$, $\nu = (\nu^{(1)}, a, 1, \nu^{(3)})$ be fixed ordered partitions of $n$ where

- $\eta^{(1)}$, $\eta^{(2)}$, $\nu^{(1)}$, $\nu^{(3)}$ are ordered partitions.
- $x, a \in \mathbb{N}_0$
- $\eta^{(1)} \lhd \nu^{(1)}$.
- $\nu^{(3)}$ consists entirely of ones.
- $x + a > n - \sum \nu^{(1)}$.

Then we define $\eta_1 = (\eta^{(1)}, \eta^{(2)}, x - n + \sum \nu^{(1)} + a, n - \sum \nu^{(1)} - a)$, $\nu_1 = (\nu^{(1)}, a + 1, \nu^{(3)})$. To simplify the notations we write $K = Q_\eta$, $K_1 = Q_{\eta_1}$, $L = Q_\nu$, $L_1 = Q_{\nu_1}$ and $\mathcal{W} = S_K$, $\mathcal{W}_1 = S_{K_1}$

**Lemma 7.2.1.**

$$\forall w \in \mathcal{W} \setminus \mathcal{W}_1 : \exists w' \in \mathcal{W}_1 : wL^\circ \cap L^\circ \subset w' L_1^\circ$$

*Proof.* We will denote the position of $a$ in $\nu$ by $\alpha$. The beginning of $\eta^{(2)}$ in $\eta$ will be at position $\beta'$ and $x$ will be at position $\beta$.

First we make a few remarks which follow either from the definitions or else by counting.

(1) $\bigcup_{i<\alpha} L_i = \bigcup_{i<\beta'} K_i$ and the second decomposition is a refinement of the first.
(2) $\bigcup_{i \geq \alpha+1} L_i = K_{1,\beta+1}$
(3) $L_\alpha = \bigcup_{\beta' \leq i \leq \beta} K_{1,i}$
(4) $\mathcal{W} \setminus \mathcal{W}_1 = \prod_{i \neq \beta} S_{K_i} \times (S_{K_\beta} \setminus (S_{K_{1,\beta}} \times S_{K_{1,\beta+1}}))$
(5) $L_1^\circ = L^\circ \setminus (L_\alpha \times L_{\alpha+1})$

Now we will try to bound the sets $wL^\circ \cap L^\circ$ where $w \in \mathcal{W} \setminus \mathcal{W}_1$. To this end we compute

$$\begin{aligned} L^\circ &= \Big(\bigcup_{\substack{i<j \\ i<\alpha}} L_i \times L_j\Big) \cup \Big(\bigcup_{\substack{i<j \\ i\geq\alpha}} L_i \times L_j\Big) \\ &\subset \Big(\bigcup_{\substack{i<j \\ i<\alpha}} L_i \times L_j\Big) \cup \left[\Big(\bigcup_{i\geq\alpha} L_i\Big) \times \Big(\bigcup_{j\geq\alpha+1} L_j\Big)\right] \end{aligned}$$

Using 1. we see that

$$\bigcup_{\substack{j>i\\ i<\alpha}} L_i \times L_j$$

and $\bigcup_{i\geq\alpha} L_i$ are $\mathcal{W}$-invariant.

Furthermore by 2. and 4. $\bigcup_{j\geq\alpha+1} L_j$ cannot be $\mathcal{W}$-invariant if $w \in \mathcal{W}\setminus\mathcal{W}_1$. Hence there exists a $p$ in $\bigcup_{j\geq\alpha+1} L_j$ such that

$$L^\circ \cup wL^\circ \subset (\bigcup_{\substack{i<\alpha\\ j>i}} L_i \times L_j) \cup \left[(\bigcup_{\substack{i\geq\alpha\\ j>i}} L_i \times L_j) \setminus (\bigcup_{i\geq\alpha} L_i \times \{p\})\right] \tag{31}$$

We will now show that the right hand side of (31) is contained in some $w'L_1^\circ$ for $w' \in \mathcal{W}_1$. Assume that $L_{\alpha+1} = \{q\}$. We define $w' = (p, p-1, \ldots, q)$. By 2. we see that $w' \in \mathcal{W}_1$. We will now decompose $L_1^\circ$ (using 5.).

$$L_1^\circ = (\bigcup_{\substack{i<\alpha\\ j>i}} L_i \times L_j) \cup \left[(\bigcup_{\substack{i=\alpha\\ j>i}} L_i \times L_j) \setminus (L_\alpha \times \{q\})\right] \cup (\bigcup_{\substack{i>\alpha\\ j>i}} L_i \times L_j)$$

Here

$$\bigcup_{\substack{i>\alpha\\ j>i}} L_i \times L_j = \{(p_1, p_2) \mid p_1, p_2 \in Q, p_2 > p_1 \geq q\}$$

Therefore

$$w'(\bigcup_{\substack{i>\alpha\\ j>i}} L_i \times L_j) \supset (\bigcup_{\substack{i>\alpha\\ j>i}} L_i \times L_j) \setminus (\bigcup_{i>\alpha} L_i \times \{p\}) \tag{32}$$

(This is the key point.)

Using 2., 3. and (32) we deduce that

$$\begin{aligned} w'L'^\circ \;\supset\; & (\bigcup_{\substack{i<\alpha\\ j>i}} L_i \times L_j) \cup \left[(\bigcup_{\substack{i=\alpha\\ j>i}} L_i \times L_j) \setminus (L_\alpha \times \{p\})\right] \cup \left[(\bigcup_{\substack{i>\alpha\\ j>i}} L_i \times L_j) \setminus (\bigcup_{i>\alpha} L_i \times \{p\})\right] \\ =\; & (\bigcup_{\substack{i<\alpha\\ j>i}} L_i \times L_j) \cup \left[(\bigcup_{\substack{i\geq\alpha\\ j>i}} L_i \times L_j) \setminus (\bigcup_{i\geq\alpha} L_i \times \{p\})\right] \end{aligned}$$

which is precisely the right hand side of (31). □

**Corollary 7.2.2.** *$(G(\lambda_{\eta_1}), X_{\lambda_{\nu_1}})$ is a reduction pair for $(G(\lambda_\eta), X_{\lambda_\nu})$*

*Proof.* Immediate from lemma 7.1.3 and lemma 7.2.1. □

7.3. **Good pairs of ordered partitions.** In this subsection we will prove that for certain pairs of partitions $(\eta, \nu)$, the pair $(G(\lambda_\eta), X_{\lambda_\nu})$ is constructible. We will build upon the result obtained in Cor. 7.2.2.

**Definition 7.3.1.** Let $(\eta, \nu)$ be a pair of ordered partitions of $n$. Then we say that $(\eta, \nu)$ is good if one of the following holds :

(1) $\eta \lhd \nu$
(2) $\eta = (\eta^{(1)}, \eta^{(2)}, x)$, $\nu = (\nu^{(1)}, a, \nu^{(2)})$ where
  (a) $\eta^{(1)}$, $\eta^{(2)}$, $\nu^{(1)}$, $\nu^{(2)}$ are ordered partitions.
  (b) $x$, $a \in \mathbb{N}_0$
  (c) $\eta^{(1)} \lhd \nu^{(1)}$
  (d) $\nu^{(2)}$ consists entirely of ones.
  (e) $a + x > n - \sum \nu^{(1)}$

**Lemma 7.3.2.** *Assume that* $\eta = (\eta^{(1)}, \eta^{(2)}, x)$ *and* $\nu = (\nu^{(1)}, a)$ *are ordered partitions of* $n$ *such that* $\eta^{(1)} \lhd \nu^{(1)}$. *Then* $\eta \lhd \nu$.

*Proof.* Clear. □

**Lemma 7.3.3.** *Let* $\eta = (\eta^{(1)}, \eta^{(2)}, x)$ *and* $\nu = (\nu^{(1)}, a, \nu^{(2)})$ *be ordered partitions of* $n$ *such that* $\eta^{(1)} \lhd \nu^{(1)}$ *and* $\nu^{(2)}$ *consists entirely of ones. Assume that* $a + x = n - \sum \nu^{(1)}$. *Then the pair* $(\eta, \nu)$ *is good.*

*Proof.* By lemma 7.3.2 we may assume that $\nu^{(2)} = (1, \nu^{(3)})$. We deduce $a + x = n - \sum \nu^{(1)} = n - \sum \eta^{(1)} = \sum \eta^{(2)} + x$ or $\sum \eta^{(2)} = a$. Hence $\eta^{(3)} = (\eta^{(1)}, \eta^{(2)}) \lhd (\nu^{(1)}, a) = \nu^{(4)}$. We then rewrite $\eta = (\eta^{(3)}, x)$, $\nu = (\nu^{(4)}, 1, \nu^{(3)})$. Now one sees that $(\eta, \nu)$ satisfies the first four conditions of 7.3.1.2 (making the appropriate translations).

To check condition 2e. we observe that $1 + x > n - \sum \nu^{(4)}$ since $n - \sum \nu^{(4)} = n - \sum \nu^{(1)} - a = x$. □

Assume now that we have a good pair of partitions $(\eta, \nu)$, but not $\eta \lhd \nu$. Then $\eta = (\eta^{(1)}, \eta^{(2)}, x)$, $\nu = (\nu^{(1)}, a, \nu^{(2)})$ as in definition 7.3.1. Furthermore by lemma 7.3.2 $\nu^{(2)}$ is non empty and hence $\nu^{(2)} = (1, \nu^{(3)})$. We can then define $(\eta_1, \nu_1)$ as in 7.2, i.e. $\eta_1 = (\eta^{(1)}, \eta^{(2)}, x - n + \sum \nu^{(1)} + a, n - \sum \nu^{(1)} - a)$ and $\nu_1 = (\nu^{(1)}, a + 1, \nu^{(2)})$.

**Lemma 7.3.4.** $(\eta, \nu_1)$*,* $(\eta_1, \nu)$ *and* $(\eta_1, \nu_1)$ *are good.*

*Proof.*

- $(\eta, \nu_1)$ is good because $a + 1 + x > n - \sum \nu^{(1)}$ (using Def. 7.3.1).
- $(\eta_1, \nu)$ is good because $n - \sum \nu^{(1)} - a + a = n - \sum \nu^{(1)}$. Hence we can apply lemma 7.3.3.
- $(\eta_1, \nu_1)$ is good because $n - \sum \nu^{(1)} - a + a + 1 > n - \sum \nu^{(1)}$.

□

**Corollary 7.3.5.** *If* $(\eta, \nu)$ *is a good pair of ordered partitions of* $n$ *then* $(G(\lambda_\eta), X_{\lambda_\nu})$ *is constructible.*

*Proof.* Clear from 7.2.2, 7.3.4 and the fact that eventually 1. of Def. 7.3.1 must become true. □

**Theorem 7.3.6.** $\mathbb{T}_{m,n}$ *is Cohen Macaulay for all* $(m, n)$.

*Proof.* As we have seen before (3) $\mathbb{T}_{m,n} = (\mathrm{End}(V) \otimes k[W])^G$. The case $n = 1$ is trivial. Furthermore it is easily verified that $\mathbb{T}_{1,n}$ is a free module over its center which is a polynomial ring. Hence the result is clear. $\mathbb{T}_{2,2}$ was treated in [5][18].

Hence it remains to consider the cases $(m, n) \geq (2, 3)$ and $\geq (3, 2)$.

It is well known that $\mathrm{End}(V) = k \oplus \mathrm{End}(V)^0$ where the elements of $\mathrm{End}(V)^0$ are those endomorphisms of $V$ having trace 0. This is an irreducible $G$-representation. Since $k[W]^G$ is Cohen-Macaulay by the Hochster Roberts theorem, it suffices to look at the case $U = \mathrm{End}(V)^0$. It is easy to verify that the character of $U^*$ is strongly critical for $(G, W)$ if $(m, n) \geq (2, 3)$ or $\geq (3, 2)$. Furthermore, if $m \geq 2$ then the action on $X$ is generically free and hence $X$ has a stable point by 6.1.8. Hence the only thing that has to be proved, to apply Th. 6.1.9, is that $X^u$ is constructible. However by 7.1.2 $X^u = G(\lambda_{\eta_{\max}})X_{\lambda_{\eta_{\min}}}$ and according to Def. 7.3.1 $(\eta_{\max}, \eta_{\min})$ is a good pair of ordered partitions ($\eta^{(1)}$, $\eta^{(2)}$ and $\nu^{(1)}$ are empty in this case). Hence we may apply Cor. 7.3.5. □

*Remark* 7.3.7. It may be somewhat surprising that the cases $m = 1$ and $(m, n) = (2, 2)$ play a special role in the above argument. However it is easily verified that

$$P(\mathbb{T}_{1,n}, t) = \frac{1}{(1-t)^2(1-t^2)\cdots(1-t^{n-1})}$$

and

$$P(\mathbb{T}_{2,2}, t) = \frac{1}{(1-t)^4(1-t^2)}$$

Hence they satisfy the functional equations

$$P(\mathbb{T}_{1,n}, 1/t) = (-1)^n t^{\frac{n^2-n+2}{2}} P(\mathbb{T}_{1,n}, t)$$

and

$$P(\mathbb{T}_{2,2}, 1/t) = -t^6 P(\mathbb{T}_{2,2}, t)$$

which are different from the functional equation for $(m, n) \geq (2, 3)$ or $\geq (3, 2)$ (as predicted by (4))

$$P(\mathbb{T}_{m,n}, 1/t) = (-1)^{(m-1)n^2+1} t^{mn^2} P(\mathbb{T}_{m,n}, t)$$

For other proofs of this functional equation we refer to [4][10][21]. However these authors seem to have been unaware of the general result in [20].

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