A CONVERSE TO STANLEY'S CONJECTURE FOR Sl_2.

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ABSTRACT. In this note we prove, in the case of Sl_2, a converse to Stanley's conjecture about Cohen-Macaulayness of invariant modules for reductive algebraic groups.

1. Introduction

Let $G = \mathrm{Sl}(V)$ where V is a twodimensional vectorspace over an algebraically closed field k of characteristic zero. Define $W = \oplus_{i=1}^m S^{d_i}V$, $d = \dim W = \sum(d_i+1)$, $R = SW$, where SW denotes the symmetric algebra of W.

Define for $n \geq 0$

$$s^{(n)} = \begin{cases} n + (n-2) + \cdots + 1 = \dfrac{(n+1)^2}{4} & \text{if } n \text{ is odd} \\ n + (n-2) + \cdots + 2 = \dfrac{n(n+2)}{4} & \text{if } n \text{ is even} \end{cases}$$

and put $s = \sum_{i=1}^m s^{(d_i)}$.

It follows from a conjecture of Stanley [2] that $(R \otimes S^\mu V)^G$ is Cohen-Macaulay if $\mu < s - 2$. This conjecture was proved partially in [5], and in almost complete generality in [4].

In [1] B. Broer proved a partial converse to Stanley's conjecture for Sl_2. In this note we will prove a complete converse.

We may always drop all trivial irreducible components of W since the Cohen-Macaulayness of $(R \otimes S^\mu V)^G$ is not affected by them. Hence we assume from now that all $d_i > 0$. We separate the following cases :

(A) $W = V$, S^2V, $V \oplus V$, $V \oplus S^2V$, $S^2V \oplus S^2V$, S^3V, S^4V.
(B) All d_i are even and u is odd.
(C) All other cases.

In this note we will prove the following theorem :

Theorem 1.1. *In case* (A) $(R \otimes S^\mu V)^G$ *is always Cohen-Macaulay. In case* (B) $(R \otimes S^\mu V)^G = 0$*. In case* (C) *the converse to Stanley's conjecture is true.*

It should be noted that, in connection with a possible converse to Stanley's conjecture, one cannot expect a nice, succinct statement. See e.g. [3], and in particular Example 4.5, for the torus case.

Case (B) of Theorem 1.1 is easy to see by looking at the action of the center of G on $(R \otimes S^\mu V)^G$.

The representations listed in case (A) are the so-called "equidimensional" representations. I.e. those for which the quotient map $R \to R^G$ is equidimensional.

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It is well-known that this implies that all $(R \otimes S^{\mu}V)^G$ are Cohen-Macaulay. One possible argument is given in the beginning of the next section.

The reader should note however that more is true. Namely, in case (A), R^G turns out to be always a polynomial ring. This is a special case of the "Russian conjecture" which remains open for general reductive groups. Hence in case (A) all $(R \otimes S^{\mu}V)^G$ are actually free.

2. The method

Keep the same notations as above. In the sequel $R = SW$ will be equipped with its natural $\mathbb{Z}$-grading. Let $I = R(R^G)^+$, $h = \dim R^G$. Recall from [4] that $(R \otimes S^{\mu}V)^G$ is Cohen-Macaulay if and only if $S^{\mu}V$ does not occur as a summand when $H^i_I(R)$ for $i = 0, \ldots, h-1$ is decomposed as a sum of irreducible representation of G.

Let $X = \operatorname{Spec} R$. The radical of I is the defining ideal of the G-unstable locus in X, which will be denoted by X^u. I.e.

$$X^u = \{x \in X \mid 0 \in \overline{Gx}\}$$

In particular $H^i_I(R) = H^i_{X^u}(X, \mathcal{O}_X)$ and

$$H^i_{X^u}(X, \mathcal{O}_X) = 0 \quad \text{for} \quad 0 < i < \operatorname{codim}(X_u, X) \tag{1}$$

Fix a basis for V and use this basis to identify $\mathrm{Sl}(V)$ with $\mathrm{Sl}_2(k)$. Let $z \mapsto \operatorname{diag}(z, z^{-1})$ be a one parameter subgroup of G, and let

$$X_\lambda = \{x \in X \mid \lim_{t \to 0} \lambda(t)x = 0\}$$

Then it follows from the Hilbert Mumford criterion that $X^u = GX_\lambda$. Hence we have to compute $H^i_{GX_\lambda}(X, \mathcal{O}_X)$ for $0 \leq i < h$.

Let $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ be resp. a Borel subgroup and a maximal torus in G. Then B acts on X_λ and it is easy to verify that the standard map $G \times^B (X_\lambda - \{0\}) \to GX_\lambda - \{0\}$ is settheoretically a bijection.

Hence $\dim X^u = 1 + \dim X_\lambda$. Therefore using (1) we find that if $1 + \dim X_\lambda + h \leq d$ then all $(R \otimes S^{\mu}V)^G$ are Cohen-Macaulay. An easy verifcation shows that this is precisely the case for the representations in (A).

Having settled cases (A) and (B) we now concentrate on the proof of (C).

Let $[e] \in G/B$ be the class of the unit element. Taking the fiber over $[e]$ defines an equivalence between $\mathcal{O}_{G \times^B X}$-modules with a G-action and $\mathcal{O}_X$-modules with a B-action. The inverse of this functor will be denoted by $\tilde{}$.

Assume that W is not V or S^2V. (These cases are included in (A).) In that case X has a G-stable point and hence $h = d - 3$. There is a long exact sequence

$$H^i_{\{0\}}(X, \mathcal{O}_X) \to H^i_{GX_\lambda}(X, \mathcal{O}_X) \to H^i_{GX_\lambda - \{0\}}(X - \{0\}, \mathcal{O}_X) \to H^{i+1}_{\{0\}}(X, \mathcal{O}_X)$$

But $H^{i(+1)}_{\{0\}}(X, \mathcal{O}_X) = 0$ if $i(+1) \neq d$. Hence it suffices to compute

$$H^i_{GX_\lambda - \{0\}}(X - \{0\}, \mathcal{O}_X) \text{ for } 0 \leq i < d - 3$$

Using [5, lem. 3.2], together with the definition of algebraic De Rham homology we obtain that

$$H^{i-2}_{GX_\lambda - \{0\}}(X - \{0\}, \mathcal{O}_X) = \mathbb{H}^i_{G \times (X_\lambda - \{0\})}(G \times^B X, \Omega^\cdot)$$

Here $\Omega^\cdot$ denotes the relative De Rham complex of $G \times^B X/X$ and $\mathbb{H}^*_?$ denotes hypercohomology with support. Hence we obtain a spectral sequence

$$E_1^{pq} : H^q_{G\times^B(X_\lambda-\{0\})}(G \times^B (X - \{0\}), \wedge^p\Omega) \Rightarrow H^{p+q-2}_{GX_\lambda-\{0\}}(X - \{0\}, \mathcal{O}_X)$$

First note that $E_1^{pq} = 0$ unless $p = 0, 1$. We will compute the terms in this spectral sequence under the hypothesis

$$p + q - 2 < d - 3 \tag{2}$$

There is a long exact sequence

$$\cdots \to H^q_{G\times^B\{0\}}(G \times^B X, \wedge^p\Omega) \to H^q_{G\times^B X_\lambda}(G \times^B X, \wedge^p\Omega)$$
$$\to H^q_{G\times^B(X_\lambda-\{0\})}(G \times^B (X - \{0\}), \wedge^p\Omega) \to H^{q+1}_{G\times^B\{0\}}(G \times^B X, \wedge^p\Omega) \to \cdots$$

But $H^{q(+1)}_{G\times^B\{0\}}(G \times^B X, \wedge^p\Omega) = 0$ unless $q(+1) \geq d$.

Hence under hyp. (2)

$$E_1^{pq} = H^q_{G\times^B X_\lambda}(G \times^B X, \wedge^p\Omega)$$

We now employ the composite functor spectral sequence

$$E_2^{q'q''} : H^{q'}(\mathcal{H}^{q''}_{G\times^B X_\lambda}(G \times^B X, \wedge^p\Omega)) \Rightarrow H^{q'+q''}_{G\times^B X_\lambda}(G \times^B X, \wedge^p\Omega)$$

$G \times^B X_\lambda$ is a local complete intersection in $G \times^B X$ and hence $\mathcal{H}^{q''}_{G\times^B X_\lambda}(G \times^B X, \wedge^p\Omega) = 0$ unless $q'' = d_\lambda$ where $d_\lambda = \operatorname{codim}(X_\lambda, X) = \sum_{i=1,\dots,m} \left\lceil \frac{d_i+1}{2} \right\rceil$.

Furthermore

$$\mathcal{H}^{d_\lambda}_{G\times^B X_\lambda}(G \times^B X, \wedge^p\Omega) = H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)\tilde{\ } \otimes_{\mathcal{O}_{G/B}} \wedge^p\Omega_{G/B}$$

Put $Z = H^{d_\lambda}_{X_\lambda}(X, \mathcal{O}_X)$. Then we obtain

$$E_1^{pq} = H^{q-d_\lambda}(G/B, \tilde{Z} \otimes \wedge^p\Omega_{G/B})$$

Hence (still under hyp. (2)) $E_1^{pq} = 0$ unless $q = d_\lambda$, $d_\lambda + 1$ and $p = 0, 1$. For simplicity we put

$$A_{i,j} = H^j(G/B, \tilde{Z} \otimes \wedge^i\Omega_{G/B})$$

To estimate $A_{i,j}$ we define Z' to be the B-representation on which the unipotent part of B acts trivially but which has the same T-weights as Z. $A'_{i,j}$ will be defined as $A_{i,j}$ but with Z replaced by Z'.

Let $\chi : \operatorname{diag}(z, z^{-1}) \mapsto z$ be the generator of $X(T)$ and let $(\chi^{u_i})_{i=1,\dots,d}$ be the T-weights of W. Then the T-weights of Z are [3]

$$\chi^{-\sum_{u_i\geq 0}(a_i+1)u_i + \sum_{u_i<0} b_i u_i} \tag{3}$$

where $(a_i)_i$, $(b_i)_i \in \mathbb{N}$, and such a weight occurs in degree

$$\sum b_i - \sum(a_i + 1)$$

Now note that $G/B \cong \mathbb{P}^1$. We claim that $\tilde{\chi} = \mathcal{O}(-1)$, or equivalently $\chi = \mathcal{O}(-1)_e$ where e is the fixpoint for the B-action on $\mathbb{P}^1$. Then $\mathcal{O}(-1) = \mathcal{O}(-e)$, and hence $\mathcal{O}(-1)_e \cong m_e/m_e^2$ with m_e the maximal ideal of $\mathcal{O}_{\mathbb{P}^1,e}$. A local computation now shows what we want.

Lemma 2.1. $A_{i,1} = 0$

Proof. Z is a rational representation of B and therefore we may construct a left limited ascending filtration on Z such that $\operatorname{gr} Z = Z'$. Hence it suffices to prove the lemma for $A'_{i,1}$. By the above we have to show that

$$H^1(G/B, \mathcal{O}(\sum_{u_i \geq 0}(a_1+1)u_i - \sum_{u_i<0} b_i u_i - 2))$$
$$= H^0(G/B, \mathcal{O}(-\sum_{u_i\geq 0}(a_i+1)u_1 + \sum_{u_i<0} b_i u_i)) = 0$$

It is clear that this is always the case. □

Lemma 2.2.

(1) *The arrow from position $(0, d_\lambda)$ to position $(1, d_\lambda)$ in E_1 is injective.*
(2) *The position $(1, d_\lambda)$ lies strictly below the line $p+q-2 = d-3$ if and only if we are not in case (A).*

Proof. (1) This follows from $\operatorname{codim}(X^u, X) = d_\lambda - 1$ and hence $H^i_{X^u}(X, \mathcal{O}_X) = 0$ if $i < d_\lambda - 1$. If the arrow were not injective then $H^{d_\lambda-2}_{X^u}(X, \mathcal{O}_X) \neq 0$.
(2) This is a simple verification. □

Assume that U is a $\mathbb{Z}$-graded G-representation. We define

$$P(U, x, t) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \operatorname{Mult}_{S^r V}(U_s) x^r t^s$$

In the sequel such an expression is supposed to define an element of $k((t^{-1}))[[x]]$. Let e be the number of even d_i's.

Lemma 2.3.

$$P(A_{0,0}, x, t) = \frac{t^{-d_\lambda}}{(1-t^{-1})^e} x^s \frac{1}{\prod_{u_i>0}(1-x^{u_i}t^{-1})\prod_{u_i<0}(1-x^{-u_i}t)} \quad (4)$$

$$P(A_{1,0}, x, t) = \frac{t^{-d_\lambda}}{(1-t^{-1})^e} x^{s-2} \frac{1}{\prod_{u_i>0}(1-x^{u_i}t^{-1})\prod_{u_i<0}(1-x^{-u_i}t)} \quad (5)$$

Proof. Since $A'_{0,1} = 0$ it is easy to see that $P(A_{0,0}, x, t) = P(A'_{0,0}, x, t)$. From (3) it follows that

$$P(A'_{0,0}, x, t) = \sum_{(a_i),(b_i)} x^{\left(\sum_{u_i\geq 0}(a_i+1)u_i - \sum_{u_i<0} b_i u_i\right)} t^{\left(\sum_{u_i<0} b_i - \sum_{u_i\geq 0}(a_i+1)\right)}$$

which evaluates to the righthand side of (4).

The proof for (5) is similar. □

We are now ready to prove the following theorem :

Theorem 2.4. *Assume that we are not in case (A). Then $H^i_I(R) = 0$ unless $i = d_\lambda - 1,\ d-3$.*

Furthermore

$$P(H_I^{d_\lambda-1}(R), x, t) = \frac{t^{-d_\lambda}}{(1-t^{-1})^e} x^{s-2} \frac{1-x^2}{\prod_{u_i>0}(1-x^{u_i}t^{-1})\prod_{u_i<0}(1-x^{-u_i}t)}$$

Proof. That $H^i_I(R) = 0$ unless $i = d_\lambda - 1$, $d - 3$ follows from lemmas 2.1, 2.2. The statement about the Poincare series follows from the fact that there is an exact sequence

$$0 \to A_{0,0} \to A_{1,0} \to H_I^{d_\lambda - 1}(R) \to 0$$

and hence

$$P(H_I^{d_\lambda - 1}(R), x, t) = P(A_{1,0}, x, t) - P(A_{0,0}, x, t)$$

We then apply lemma 2.3. □

Proof of Theorem 1.1. It is easy to see that all powers of x appear in the expansion of

$$\frac{1 - x^2}{\prod_{u_i > 0}(1 - x^{u_i} t^{-1}) \prod_{u_i < 0}(1 - x^{-u_i} t)}$$

unless all d_i are even. In that case all even powers appear. □

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