A CONVERSE TO STANLEY’S CONJECTURE FOR $\text{Sl}_2$.

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Abstract. In this note we prove, in the case of $\text{Sl}_2$, a converse to Stanley’s conjecture about Cohen-Macaulayness of invariant modules for reductive algebraic groups.

1. Introduction

Let $G = \text{Sl}(V)$ where $V$ is a twodimensional vectorspace over an algebraically closed field $k$ of characteristic zero. Define $W = \bigoplus_{i=1}^{m} S^{d_i}V$, $d = \dim W = \sum (d_i+1)$, $R = SW$, where $SW$ denotes the symmetric algebra of $W$.

Define for $n \geq 0$

$$s^{(n)} = \begin{cases} n + (n-2) + \cdots + 1 = \frac{(n+1)^2}{4} & \text{if } n \text{ is odd} \\ n + (n-2) + \cdots + 2 = \frac{n(n+2)}{4} & \text{if } n \text{ is even} \end{cases}$$

and put $s = \sum_{i=1}^{m} s^{(d_i)}$.

It follows from a conjecture of Stanley [2] that $(R \otimes S^{\mu}V)^G$ is Cohen-Macaulay if $\mu < s - 2$. This conjecture was proved partially in [5], and in almost complete generality in [4].

In [1] B. Broer proved a partial converse to Stanley’s conjecture for $\text{Sl}_2$. In this note we will prove a complete converse.

We may always drop all trivial irreducible components of $W$ since the Cohen-Macaulayness of $(R \otimes S^{\mu}V)^G$ is not affected by them. Hence we assume from now that all $d_i > 0$. We separate the following cases:

(A) $W = V, S^2V, V \oplus V, V \oplus S^2V, S^2V \oplus S^2V, S^3V, S^4V$.

(B) All $d_i$ are even and $u$ is odd.

(C) All other cases.

In this note we will prove the following theorem:

Theorem 1.1. In case (A) $(R \otimes S^{\mu}V)^G$ is always Cohen-Macaulay. In case (B) $(R \otimes S^{\mu}V)^G = 0$. In case (C) the converse to Stanley’s conjecture is true.

It should be noted that, in connection with a possible converse to Stanley’s conjecture, one cannot expect a nice, succinct statement. See e.g. [3], and in particular Example 4.5, for the torus case.

Case (B) of Theorem 1.1 is easy to see by looking at the action of the center of $G$ on $(R \otimes S^{\mu}V)^G$.

The representations listed in case (A) are the so-called “equidimensional” representations. I.e. those for which the quotient map $R \twoheadrightarrow R^{G}$ is equidimensional.

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It is well-known that this implies that all \((R \otimes S^\mu V)^G\) are Cohen-Macaulay. One possible argument is given in the beginning of the next section.

The reader should note however that more is true. Namely, in case (A), \(R^G\) turns out to be always a polynomial ring. This is a special case of the “Russian conjecture” which remains open for general reductive groups. Hence in case (A) all \((R \otimes S^\mu V)^G\) are actually free.

2. The method

Keep the same notations as above. In the sequel \(R = SW\) will be equipped with its natural \(\mathbb{Z}\)-grading. Let \(I = R(R^G)^+, h = \dim R^G\). Recall from [4] that 
\[(R \otimes S^\mu V)^G\] is Cohen-Macaulay if and only if \(S^\mu V\) does not occur as a summand when \(H^i_j(R)\) for \(i = 0, \ldots, h-1\) is decomposed as a sum of irreducible representation of \(G\).

Let \(X = \text{Spec} R\). The radical of \(I\) is the defining ideal of the \(G\)-unstable locus in \(X\), which will be denoted by \(X^u\) i.e.

\[X^u = \{x \in X \mid 0 \in \mathcal{O}_x\}\]

In particular \(H^i_j(R) = H^i_{X^u}(X, \mathcal{O}_X)\) and

\[H^i_{X^u}(X, \mathcal{O}_X) = 0 \text{ for } 0 < i < \text{codim}(X^u, X) \quad (1)\]

Fix a basis for \(V\) and use this basis to identify \(\text{Sl}(V)\) with \(\text{Sl}_2(k)\). Let \(z \mapsto \text{diag}(z, z^{-1})\) be a one parameter subgroup of \(G\), and let

\[X_\lambda = \{x \in X \mid \lim_{t \to 0} \lambda(t)x = 0\}\]

Then it follows from the Hilbert Mumford criterion that \(X^u = GX_\lambda\). Hence we have to compute \(H^i_{GX_\lambda}(X, \mathcal{O}_X)\) for \(0 \leq i < h\).

Let \(B = \left(\begin{array}{cc} * & * \\ 0 & * \end{array}\right), \ T = \left(\begin{array}{cc} * & 0 \\ 0 & * \end{array}\right)\) be resp. a Borel subgroup and a maximal torus in \(G\). Then \(B\) acts on \(X_\lambda\) and it is easy to verify that the standard map \(G \times^B (X_\lambda - \{0\}) \to GX_\lambda - \{0\}\) is settheoretically a bijection.

Hence \(\dim X^u = 1 + \dim X_\lambda\). Therefore using (1) we find that if \(1 + \dim X_\lambda + h \leq d\) then all \((R \otimes S^\mu V)^G\) are Cohen-Macaulay. An easy verification shows that this is precisely the case for the representations in (A).

Having settled cases (A) and (B) we now concentrate on the proof of (C).

Let \([e] \in G/B\) be the class of the unit element. Taking the fiber over \([e]\) defines an equivalence between \(\mathcal{O}_{G \times X}-\text{modules with a } G\)-action and \(\mathcal{O}_X\)-modules with a \(B\)-action. The inverse of this functor will be denoted by \(\tilde{\phantom{T}}\).

Assume that \(W\) is not \(V\) or \(S^2 V\). (These cases are included in (A).) In that case \(X\) has a \(G\)-stable point and hence \(h = d - 3\). There is a long exact sequence

\[H^i_{\{0\}}(X, \mathcal{O}_X) \to H^i_{GX_\lambda}(X, \mathcal{O}_X) \to H^i_{GX_\lambda-\{0\}}(X - \{0\}, \mathcal{O}_X) \to H^{i+1}_{\{0\}}(X, \mathcal{O}_X)\]

But \(H^{i+1}_{\{0\}}(X, \mathcal{O}_X) = 0\) if \(i(+1) \neq d\). Hence it suffices to compute

\[H^i_{GX_\lambda-\{0\}}(X - \{0\}, \mathcal{O}_X) \text{ for } 0 \leq i < d - 3\]

Using [5, lem. 3.2], together with the definition of algebraic De Rham homology we obtain that

\[H^{i-2}_{GX_\lambda-\{0\}}(X - \{0\}, \mathcal{O}_X) = H^i_{G \otimes (X_\lambda - \{0\})}(G \times^B X, \Omega)\]
Here $\Omega$ denotes the relative De Rham complex of $G \times B X/X$ and $H^*_T$ denotes hypercohomology with support. Hence we obtain a spectral sequence

$$E^1_{pq}: H^q_{G \times B}(X, (-q_0)(G \times B X, \pi^p \Omega)) \Rightarrow H^{p+q-2}_{G \times B}(X, (-q_0), O_X)$$

First note that $E^1_{pq} = 0$ unless $p = 0, 1$. We will compute the terms in this spectral sequence under the hypothesis

$$p + q - 2 < d - 3$$

There is a long exact sequence

$$\cdots \rightarrow H^q_{G \times B}(X, (-0_0)(G \times B X, \pi^p \Omega)) \rightarrow H^q_{G \times B}(X, (-0_0)(G \times B X, \pi^p \Omega)) \rightarrow H^q_{G \times B}(X, (-0_0)(G \times B X, \pi^p \Omega)) \rightarrow \cdots$$

But $H^q_{G \times B}(X, (-0_0)(G \times B X, \pi^p \Omega)) = 0$ unless $q(1) \geq d$.

Hence under hyp. (2)

$$E^1_{pq} = H^q_{G \times B}(X, (-0_0)(G \times B X, \pi^p \Omega))$$

We now employ the composite functor spectral sequence

$$E^2_{pq'}: H^q(\mathcal{H}^p_{G \times B}(X, (-0_0)(G \times B X, \pi^p \Omega))) \Rightarrow H^{q'}(\mathcal{H}^p_{G \times B}(X, (-0_0)(G \times B X, \pi^p \Omega)))$$

$G \times B X/\lambda$ is a local complete intersection in $G \times B X$ and hence $\mathcal{H}^p_{G \times B}(X, (-0_0)(G \times B X, \pi^p \Omega)) = 0$ unless $q'' = d_\lambda$ where $d_\lambda = \text{codim}(X, \lambda) = \sum i=1, \ldots, m \left\lfloor \frac{d_i + 1}{2} \right\rfloor$.

Furthermore

$$\mathcal{H}^d_{G \times B}(X, (-0_0)(G \times B X, \pi^p \Omega)) = H^d_{X,X}(X, O_X) \otimes O_{G/B} \pi^p \Omega_{G/B}$$

Put $Z = H^d_{X,X}(X, O_X)$. Then we obtain

$$E^q_{pq} = H^q(G/B, \tilde{Z} \otimes \pi^p \Omega_{G/B})$$

Hence (still under hyp. (2)) $E^q_{pq} = 0$ unless $q = d_\lambda$, $d_\lambda + 1$ and $p = 0, 1$. For simplicity we put

$$A_{i,j} = H^j(G/B, \tilde{Z} \otimes \pi^i \Omega_{G/B})$$

To estimate $A_{i,j}$ we define $Z'$ to be the $B$-representation on which the unipotent part of $B$ acts trivially but which has the same $T$-weights as $Z$. $A'_{i,j}$ will be defined as $A_{i,j}$ but with $Z$ replaced by $Z'$.

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To estimate $A_{i,j}$ we define $Z'$ to be the $B$-representation on which the unipotent part of $B$ acts trivially but which has the same $T$-weights as $Z$. $A'_{i,j}$ will be defined as $A_{i,j}$ but with $Z$ replaced by $Z'$.

Let $\chi: \text{diag}(z, z^{-1}) \rightarrow z$ be the generator of $X(T)$ and let $(\chi^{a_i})_{i=1, \ldots, d}$ be the $T$-weights of $W$. Then the $T$-weights of $Z$ are [3]

$$\chi^{-\sum a_i \geq 0(a_i + 1)u + \sum b_i \geq 0 b_i u_i}$$

where $(a_i), (b_i) \in \mathbb{N}$, and such a weight occurs in degree

$$\sum b_i - \sum (a_i + 1)$$

Now note that $G/B \cong \mathbb{P}^1$. We claim that $\tilde{\chi} = O(-1)_e$, or equivalently $\chi = O(-1)_e$ where $e$ is the fixpoint for the $B$-action on $\mathbb{P}^1$. Then $O(-1) = O(-e)$, and hence $O(-1)_e \cong m_e/m_e^2$ with $m_e$ the maximal ideal of $O_{\mathbb{P}^1,e}$. A local computation now shows what we want.

**Lemma 2.1.** $A_{i,1} = 0$
Assume that we are not in case (A). Then

\[ H^1(G/B, \mathcal{O}(\sum_{u_i \geq 0} (a_i + 1)u_i - \sum_{u_i < 0} b_iu_i - 2)) = H^0(G/B, \mathcal{O}(- \sum_{u_i \geq 0} (a_i + 1)u_1 + \sum_{u_i < 0} b_iu_i)) = 0 \]

It is clear that this is always the case.

\[ \square \]

**Lemma 2.2.**

1. The arrow from position \((0, d_\lambda)\) to position \((1, d_\lambda)\) in \(E_1\) is injective.
2. The position \((1, d_\lambda)\) lies strictly below the line \(p + q - 2 = d - 3\) if and only if we are not in case (A).

**Proof.**

1. This follows from \(\text{codim}(X^u, X) = d_\lambda - 1\) and hence \(H^1_{X^u}(X, \mathcal{O}_X) = 0\) if \(i < d_\lambda - 1\). If the arrow were not injective then \(H^{d_\lambda - 2}_{X^u}(X, \mathcal{O}_X) \neq 0\).
2. This is a simple verification. \(\square\)

Assume that \(U\) is a \(\mathbb{Z}\)-graded \(G\)-representation. We define

\[ P(U, x, t) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \text{Mult}_{S^1(V)}(U^t)x^rt^s \]

In the sequel such an expression is supposed to define an element of \(k((t^{-1}))[[x]]\).

Let \(e\) be the number of even \(d_i\)'s.

**Lemma 2.3.**

\[ P(A_{0,0}, x, t) = \frac{t^{-d_\lambda}}{(1-t^{-1})e}x^1 \prod_{u_i > 0} (1 - x^{u_i}t^{-1}) \prod_{u_i < 0} (1 - x^{-u_i}) \] \hspace{1cm} (4)

\[ P(A_{1,0}, x, t) = \frac{t^{-d_\lambda}}{(1-t^{-1})e}x^{s-2} \prod_{u_i > 0} (1 - x^{u_i}t^{-1}) \prod_{u_i < 0} (1 - x^{-u_i}) \] \hspace{1cm} (5)

**Proof.** Since \(A'_{0,1} = 0\) it is easy to see that \(P(A_{0,0}, x, t) = P(A'_{0,0}, x, t)\). From (3) it follows that

\[ P(A'_{0,0}, x, t) = \sum_{(a_i, b_i)} x^{(\sum_{u_i \geq 0} (a_i + 1)u_i - \sum_{u_i < 0} b_iu_i)} t^{(\sum_{u_i < 0} b_i - \sum_{u_i \geq 0} (a_i + 1))} \]

which evaluates to the righthand side of (4).

The proof for (5) is similar. \(\square\)

We are now ready to prove the following theorem:

**Theorem 2.4.** Assume that we are not in case (A). Then \(H^i_1(R) = 0\) unless \(i = d_\lambda - 1, d - 3\).

Furthermore

\[ P(H^i_1(R), x, t) = \frac{t^{-d_\lambda}}{(1-t^{-1})e}x^{s-2} \prod_{u_i > 0} (1 - x^{u_i}t^{-1}) \prod_{u_i < 0} (1 - x^{-u_i}) \]
Proof. That $H^i_I(R) = 0$ unless $i = d_\lambda - 1, d - 3$ follows from lemmas 2.1, 2.2. The statement about the Poincare series follows from the fact that there is an exact sequence

$$0 \rightarrow A_{0,0} \rightarrow A_{1,0} \rightarrow H^{d_\lambda - 1}_I(R) \rightarrow 0$$

and hence

$$P(H^{d_\lambda - 1}_I(R), x, t) = P(A_{1,0}, x, t) - P(A_{0,0}, x, t)$$

We then apply lemma 2.3.

Proof of Theorem 1.1. It is easy to see that all powers of $x$ appear in the expansion of

$$1 - x^2 \prod_{u_i > 0} (1 - x^{u_i} t^{-1}) \prod_{u_i < 0} (1 - x^{-u_i} t)$$

unless all $d_i$ are even. In that case all even powers appear.

References


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