DIFFERENTIAL OPERATORS ON SEMI-INVARIANTS FOR TORI AND WEIGHTED PROJECTIVE SPACES

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Abstract. In this paper we study rings of differential operators for modules of covariants for one-dimensional tori. In particular we analyze when they are Morita equivalent, when they are simple, and when they have finite global dimension. As a side result we obtain an extension of the Bernstein-Beilinson equivalence to weighted projective spaces.

1. Introduction

Let k be an algebraically closed field of characteristic zero and let A be a finitely generated commutative algebra over k. Let $D(A)$ denote the ring of differential operators of A. If A is regular then $D(A)$ is simple, finitely generated left and right Noetherian. If A is not regular, all of these properties can fail [1].

One particularly interesting case is when A is a ring of invariants. Let R be a polynomial ring $k[x_1, x_2, \cdots, x_n]$ and let G be a reductive group acting linearly on R. Then R^G is a finitely generated k-algebra.

If G is finite then it is relatively easy to see that $D(R^G)$ is simple, finitely generated, left and right Noetherian [7][8]. Furthermore, as T. Stafford pointed out to me, $D(R^G)$ has finite global dimension.

In an interesting paper T. Levasseur and T. Stafford showed that if G is a classical group, and R is the symmetric algebra of a classical G-representation then again R^G is simple, finitely generated left and right Noetherian [9]. No mention is made of the global dimension of $D(R^G)$.

In [11] I. Musson showed that if G is a torus then $D(R^G)$ is finitely generated left and right Noetherian. Recently G. Schwartz proved the same result for general reductive groups, provided that the action of G on R satisfies some mild conditions [12]. However, neither I. Musson nor G. Schwartz looks at the question whether $D(R^G)$ is simple or has finite global dimension. Knowing that $D(R^G)$ is simple would be important in view of Theorem 6.2.5 which says that if $D(A)$ is simple then A is Cohen-Macaulay.

Given this incomplete understanding of $D(R^G)$, I decided to analyze the first non-trivial case, namely $G = T = \mathbb{G}_m$, a one-dimensional torus. Then we obtain a grading

$$R = \bigoplus_l R_l$$

on R where

$$R_l = \{r \in R \mid z.r = z^l r\}$$

The R_l are so-called modules of semi-invariants or modules of covariants for T. They are studied in [13][15] and for general reductive groups in [14][16].

Let $D_{R^T}(R_l)$ be the ring of differential operators on R_l. (I.e. we extend our setting slightly.) It follows easily, by a slight generalization of the methods in [12], that $D_{R^T}(R_l)$ is finitely generated, left and right Noetherian.

For our results we will have to put a minor restriction on the action of T on R that can probably be circumvented with some extra work (conditions 6.1.1 and 6.2.1 of Section 6).

We define a partition $\mathbb{Z} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ where $\mathcal{B}$ is a finite set, containing 0 and we prove

Theorem 1.1. *$D_{R^T}(R_l)$ is simple if and only if $l \in \mathcal{B}$. $D_{R^T}(R_l)$ has finite global dimension if and only if $l \in \mathcal{A} \cup \mathcal{C}$. Furthermore for $l \in \mathcal{A} \cup \mathcal{C}$ all $D_{R^T}(R_l)$ are Morita equivalent. The same is true for $l \in \mathcal{B}$.*

When we specialize to $l = 0$ we obtain

Corollary 1.2. *$D(R^T)$ is simple, but has infinite global dimension.*

Hence the situation for general G is less nice than the situation for finite G.

Finally, using the definition of $\mathcal{B}$ and [13, Th 3.3 and Cor 3.4] one may restate Theorem 1.1 in the following, somewhat curious form

Theorem 1.3. *R_l is a Cohen-Macaulay R^T-module if and only if $D_{R^T}(R_l)$ is simple, if and only if $D_{R^T}(R_l)$ does not have finite global dimension.*

A result like this can probably not be expected in general, but it nevertheless illustrates the close connection between the homological properties of modules of covariants and the ring theoretic behaviour of their rings of differential operators.

We will also give a different application of our methods. Let $Y = \mathbb{P}^n$ and let $\mathcal{D}_Y$ and D_Y be respectively the sheaf and the ring of differential operators on Y. Then the famous Bernstein-Beilinson equivalence (which was proved more generally for homogeneous spaces) states

Theorem 1.4. *The functors $\Gamma(Y, -)$, $\mathcal{D}_Y \otimes_{D_Y} -$ define mutually inverse equivalences between the category of quasi-coherent sheaves of left $\mathcal{D}_Y$-modules and the category of left D_Y-modules.*

Then, using our methods, we can generalize this theorem to weighted projective spaces (Proj's of polynomial rings in which not every variable has degree one). To my knowledge there are no other examples in the literature of projective varieties where the Berstein-Beilinson equivalence holds, that are not homogeneous spaces. It should be pointed out that our proof, when specialized to $\mathbb{P}^n$, resembles somewhat [5]. However no use is made of Lie algebra actions.

Finally we also prove a result similar to Theorem 1.1 for rings of twisted differential operators on weighted projective spaces.

It should be pointed out that although we restrict ourselves in this paper to one-dimensional tori, all of our proofs are written in such a way that they can be easily extended to actions of higher dimensional tori.

2. Preliminaries

In the sequel, k will be an algebraically closed field of characteristic zero.

If A is a commutative k-algebra and M is an A-module then $D(A)$ and $D_A(M)$ will denote the rings of differential operators of A and M. If A and M are $\mathbb{Z}$-graded, then it is easy to see that this grading carries over to a $\mathbb{Z}$-grading on $D(A)$ and $D_A(M)$.

If A is a $\mathbb{Z}$-graded ring and M is an A-module then for $l \in \mathbb{Z}$ A_l and M_l will be the parts of degree l of A and M.

In the sequel we fix a set of elements $(a_i)_{i=1,\dots,n}$ in $\mathbb{Z}$. We will always make the following assumption

Condition 2.1. *There is at least one a_i different from 0.*

Let $R = k[x_1, x_2, \dots, x_n]$ be a polynomial ring in n variables. We will put a $\mathbb{Z}$-grading on R by defining $\deg x_i = a_i$. Then

$$D(R) = k[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$$

where $\partial_i = \frac{\partial}{\partial x_i}$, is $\mathbb{Z}$-graded via

$$\begin{aligned} \deg x_i &= a_i \\ \deg \partial_i &= -a_i \end{aligned}$$

Let $\pi = \sum a_i x_i \partial_i$. Then it is easily verified that for $h \in D(R)_l$

$$[\pi, h] = lh$$

or

$$\pi h = h(\pi + l) \tag{1}$$

In particular, we deduce that π lies in the center of $D(R)_0$. For $\lambda \in k$ we define

$$D^\lambda = D(R)_0 / D(R)_0(\pi - \lambda)$$

D^λ is a finitely generated k-algebra. Note that if condition 2.1 would not hold then

$$D^\lambda = \begin{cases} D(R) & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda \neq 0 \end{cases}$$

It is clear that these represent trivial cases.

If $l \in \mathbb{Z}$ then we also define

$$D^\lambda(l) = D(R)_l / (\pi - \lambda) D(R)_l \tag{2}$$

From (1) and (2) it is clear that $D^\lambda(l)$ is a D^λ-$D^{\lambda-l}$ bimodule, finitely generated on both sides.

Furthermore, there are pairings

$$i : D^\lambda(l) \otimes_{D^{\lambda-l}} D^{\lambda-l}(-l) \to D^\lambda$$
$$j : D^{\lambda-l}(-l) \otimes_{D^\lambda} D^\lambda(l) \to D^{\lambda-l}$$

induced by the multiplication in $D(R)$.

It is easy to see that

$$(D^\lambda, D^{\lambda-l}, D^\lambda(l), D^{\lambda-l}(-l), i, j) \tag{3}$$

is a Morita context between D^λ and $D^{\lambda-l}$. Furthermore, i is surjective if and only if

$$(\pi-\lambda)D(R)_0 + D(R)_l D(R)_{-l} = D(R)_0 \tag{4}$$

and j is surjective if and only if

$$(\pi-\lambda+l)D(R)_0 + D(R)_{-l} D(R)_l = D(R)_0 \tag{5}$$

Now we prove some basic facts about D^λ. For some of the results below, it will be convenient to look at rings, slightly more general than D^λ.

Let f be a homogeneous element of R. Then we define

$$D_f^\lambda = D(R_f)_0/D(R_f)_0(\pi-\lambda)$$

Proposition 2.2.
(1) *D_f^λ is a Noetherian domain and hence a Goldie ring.*
(2) *D_f^λ is a maximal order.*
(3) *The natural map*

$$D^\lambda \to D_f^\lambda \tag{6}$$

is an injection, and both rings have the same field of fractions.
(4) *Let $l \in \mathbb{Z}$ and let $Q(D^\lambda)$ denote the quotient field of D^λ. Then there is a k-algebra injection*

$$\phi : D^{\lambda-l} \to Q(D^\lambda)$$

and a D^λ-module injection

$$\psi : D^\lambda(l) \to Q(D^\lambda)$$

such that for $a \in D^\lambda(l)$, $b \in D^{\lambda-l}$

$$\psi(ab) = \psi(a)\phi(b) \tag{7}$$

I.e. $\psi(D^\lambda(l))$ becomes a $D^\lambda - \phi(D^{\lambda-l})$ fractional ideal.

Proof. We may assume that at least two of the a_i are non-zero, otherwise the result follows by direct computation. The filtration on $D(R)$ by order of differential operators is compatible with the grading on $D(R)$, and hence it induces a filtration on D^λ.

Because $\overline{\pi - \lambda} = \overline{\pi}$ is not a zero divisor in $\operatorname{gr} D(R)_0$, we deduce that

$$\operatorname{gr} D_f^\lambda = \operatorname{gr} D(R_f)_0 / \operatorname{gr} D(R_f)_0 \overline{\pi} = (\operatorname{gr} D(R)/\operatorname{gr} D(R)\overline{\pi})_{f0} \tag{8}$$

Now, under our current hypothesis, $\operatorname{gr} D(R)/\operatorname{gr} D(R)\overline{\pi}$ is clearly a Noetherian integrally closed domain, and hence the same is true for the righthand side of (8). This proves 1., 2., using [2]. To prove that (6) is an injection, it suffices to remark that

$$\operatorname{gr} D(R)/\operatorname{gr} D(R)\overline{\pi} \to (\operatorname{gr} D(R)/\operatorname{gr} D(R)\overline{\pi})_f$$

is an injection since f is obviously not a zero divisor in $\operatorname{gr} D(R)/\operatorname{gr} D(R)\overline{\pi}$.

To prove that D^λ and D_f^λ have the same ring of fractions it suffices to remark that it follows from (8) that $\operatorname{gr} D^\lambda$ and $\operatorname{gr} D_f^\lambda$ have the same quotient field. This easily yields the result that if $a \in D_f^\lambda$ then there exists $s \in D^\lambda$ such that $sa \in D^\lambda$, which is precisely what we want.

To prove the last statement, choose $u_1, \ldots, u_n \in \mathbb{Z}$ such that $\sum a_i u_i = l$ and define $h = x_1^{u_1} \cdots x_n^{u_n}$. Then we use the fact that $D^\lambda_{x_1 \cdots x_n}$ has the same quotient field as D^λ to define

$$\phi : D^{\lambda - l} \to D^\lambda_{x_1 \cdots x_n} : \overline{x} \mapsto \overline{hxh^{-1}}$$
$$\psi : D^\lambda(l) \to D^\lambda_{x_1 \cdots x_n} : \overline{y} \mapsto \overline{yh^{-1}}$$

It is clear that ϕ and ψ have the required properties. □

Besides the $\mathbb{Z}$-grading, we have defined on $D(R)$, $D(R)$ also carries a canonical $\mathbb{Z}^n$-grading, via

$$\begin{aligned} \deg x_i &= (0, \ldots, 1, \ldots, 0) \\ \deg \partial_i &= (0, \ldots, -1, \ldots, 0) \end{aligned}$$

where ± 1 occurs in the i'th position. The part of degree zero for this new grading will be denoted by D. We have that

$$D = k[\pi_1, \ldots, \pi_n]$$

where $\pi_i = x_i \partial_i$. Clearly $\pi = \sum a_i \pi_i \in D$. This $\mathbb{Z}^n$-grading is compatible with the $\mathbb{Z}$-grading and hence it goes over to $D(R)_0$ and D^λ.

If $h \in D(R)$ is of degree $(u_1, \ldots, u_n)$ then it is easy to see that

$$[\pi_i, h] = u_i h \tag{9}$$

For $u \in \mathbb{Z}$ let

$$x^{(u)} = \begin{cases} x_i^u & \text{if } u \geq 0 \\ \partial_i^{-u} & \text{if } u < 0 \end{cases}$$

Then it is easy to see that the $\mathbb{Z}^n$-grading on D is given by the decomposition

$$D(R) = \bigoplus_{u_1,\dots,u_n\in\mathbb{Z}} Dx^{(u_1)}\cdots x^{(u_n)} = \bigoplus_{u_1,\dots,u_n\in\mathbb{Z}} x_1^{(u_1)}\cdots x_n^{(u_n)}D \tag{10}$$

We define

$$\pi_i^{(u)} = x_i^{(u)}x_i^{(-u)} = \begin{cases} x_i^u\partial_i^u & \text{if } u\geq 0\\ \partial_i^{-u}x_i^{-u} & \text{if } u<0\end{cases}$$

It is easy to see that $\pi_i^{(u)}$ is given by

$$\pi_i^{(u)} = \begin{cases} \pi_i(\pi_i-1)(\pi_i-2)\dots(\pi_i-u+1) & \text{if } u\geq 0\\ (\pi_i+1)(\pi_i+2)\cdots(\pi_i-u) & \text{if } u<0\end{cases}$$

Now we construct some useful isomorphisms between the various D^λ's. Let $j\in\{1,\dots,n\}$ and put

$$\begin{aligned} b_i &= a_i \quad \text{if } i\neq j\\ b_j &= -a_j\end{aligned} \tag{11}$$

Let π' and D'^λ be defined as π and D^λ, but with the a's replaced by the b's.

Then it is easy to see that the map

$$\begin{array}{rcl} x_j & \to & -\partial_j\\ \partial_j & \to & x_j\end{array} \quad \text{and} \quad \begin{array}{rcl} x_i & \to & x_i\\ \partial_i & \to & \partial_i\end{array} \quad \text{for } i\neq j$$

induces an isomorphism between D^λ and $D'^{\lambda+a_j}$.

We obtain two useful corollaries.

Corollary 2.3. (1) *D^λ and $D^{-\lambda-\sum a_i}$ are isomorphic.*

(2) *For all i let $b_i = |a_i|$. Then D^λ and $D'^{\lambda-\sum_{a_i<0}|a_i|}$ are isomorphic.*

Hence in principle we may always assume that all a_i's are positive.

We end this preliminary section by defining some special subsets of k. At this moment these definitions may seem arbitrary, but it will become clear in the next sections that the properties of D^λ depend heavily on which of those sets λ is a member.

$$\begin{aligned}
\mathcal{E} &= k \setminus \mathbb{Z}\mathrm{gcd}(a_i)_i \\
\mathcal{E}_\mu &= \{\mu + l \,\mathrm{gcd}(a_i)_i \mid l \in \mathbb{Z}\} \quad \text{for } \mu \in \mathcal{E} \\
\mathcal{A} &= \{\lambda \in \mathrm{gcd}(a_i)_i\mathbb{Z} \mid \exists (\alpha_i)_i \in \mathbb{Z}^n : \sum a_i\alpha_i = \lambda \text{ and} \\
&\qquad a_i \leq 0 \Rightarrow \alpha_i < 0 \\
&\qquad a_i > 0 \Rightarrow \alpha_i \geq 0\} \\
\mathcal{C} &= \{\lambda \in \mathrm{gcd}(a_i)_i\mathbb{Z} \mid \exists (\alpha_i)_i \in \mathbb{Z}^n : \sum a_i\alpha_i = \lambda \text{ and} \\
&\qquad a_i \geq 0 \Rightarrow \alpha_i < 0 \\
&\qquad a_i < 0 \Rightarrow \alpha_i \geq 0\} \\
\mathcal{B} &= k \setminus (\mathcal{A} \cup \mathcal{C})
\end{aligned}$$

Example 2.4. Assume that all a_i are equal to one. Then

$$\begin{aligned}
\mathcal{E} &= k \setminus \mathbb{Z} \\
\mathcal{A} &= \mathbb{N} \\
\mathcal{C} &= \{l \in \mathbb{Z} \mid l \leq -n\} \\
\mathcal{B} &= \{-1, -2, \ldots, -n+1\}
\end{aligned}$$

Lemma 2.5. (1) *$\mathcal{A} \geq -\sum_{a_i<0} a_i$ and $\mathcal{C} \leq -\sum_{a_i>0} a_i$.*

(2) *$\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}\}$ is a decomposition in pairwise disjoint subsets of k.*

(3) *If there are at least two non-zero a_i's then $\mathcal{B}$ is non-empty.*

(4) *If there are at least two non-zero a_i's with different signs then $0 \in \mathcal{B}$.*

(5) *$\mathcal{B}$ is a finite set*

Proof. 1. follows from the definitions of $\mathcal{A}$ and $\mathcal{C}$. 2. follows from 1. and condition 2.1. For 3. let $a_{i_1}a_{i_2} \neq 0$. Put

$$b = \begin{cases} -|a_{i_1} - a_{i_2}| & \text{if} \quad a_{i_1} > 0,\, a_{i_2} > 0 \\ 0 & \text{if} \quad a_{i_1}a_{i_2} < 0 \\ |a_{i_1} - a_{i_2}| & \text{if} \quad a_{i_1} < 0,\, a_{i_2} < 0 \end{cases}$$

Then $-\sum_{a_i>0} a_i < b < -\sum_{a_i<0} a_i$ and hence $b \in B$.

4. is part of the proof of 3. To prove 5. it suffices to remark that for $t \gg 0$

$$\begin{aligned}
\mathcal{A} &\supset |\mathrm{gcd}(a_i)_i|\{i \in \mathbb{Z} \mid t \geq t_0\} \\
\mathcal{C} &\supset |\mathrm{gcd}(a_i)_i|\{i \in \mathbb{Z} \mid t \leq -t_0\}
\end{aligned}$$

□

Lemma 2.6. *Let $(b_i)_i \in \mathbb{Z}^n$ and let $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$, $\mathcal{E}'$, $\mathcal{E}'_\mu$ be defined as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$, $\mathcal{E}_\mu$ but using the $(b_i)_i$ instead of the $(a_i)_i$.*

(1) *Let* $j \in \{1, \ldots, n\}$ *and let* $(b_i)_i$ *be defined as in (11). Then*

$$\begin{aligned} \mathcal{E}'_\mu &= \mathcal{E}_\mu \\ \mathcal{A}' &= \mathcal{A} + a_j \\ \mathcal{B}' &= \mathcal{B} + a_j \\ \mathcal{C}' &= \mathcal{C} + a_j \end{aligned}$$

(2)

$$\begin{aligned} \mathcal{E}_\mu &= -\mathcal{E}_{-\mu} \\ \mathcal{A} &= -\mathcal{C} - \sum_i a_i \\ \mathcal{B} &= -\mathcal{B} - \sum_i a_i \\ \mathcal{C} &= -\mathcal{A} - \sum_i a_i \end{aligned}$$

(3) *Let* $b_i = |a_i|$ *for all* i. *Then*

$$\begin{aligned} \mathcal{E}_\mu &= \mathcal{E}_\mu \\ \mathcal{A}' &= \mathcal{A} - \sum_{a_i<0} |a_i| \\ \mathcal{B}' &= \mathcal{B} - \sum_{a_i<0} |a_i| \\ \mathcal{C}' &= \mathcal{C} - \sum_{a_i<0} |a_i| \end{aligned}$$

Proof. 2. and 3. follow by repeated application of 1. which is left to the reader. □

3. The simplicity of D^λ

In this section we will use the same notations as in the previous section.

From (9) it follows that any two-sided ideal in $D(R)_0$ must be $\mathbb{Z}^n$-graded. Furthermore, since D^λ is a domain (prop. 2.2) and since the part of degree $(u_1,\dots,u_n)$ of D^λ is non-zero if and only if the part of degree $(-u_1,\dots,-u_n)$ is non-zero (using the involution $\overline{x}_i \to \overline{\partial}_i$, $\overline{\partial}_i \to \overline{x}_i$ on $\operatorname{gr} D^\lambda$) it is easy to see that any non-zero ideal in D^λ intersects non-trivially the part of degree zero of D^λ, which is $D/D(\pi-\lambda)$.

Using (10) this leads to the following statement :

D^λ is simple if and only if for all $f \in D$, not divisible by $\pi - \lambda$

$$(\pi-\lambda)D + \sum_{\substack{(u_1,\dots,u_n)\in\mathbb{Z}^n \\ \sum a_i u_i = 0}} x_1^{(u_1)}\cdots x_n^{(u_n)} f x_1^{(-u_1)}\cdots x_n^{(-u_n)} D = D$$

Now it is easy to see that

$$x_1^{(u_1)}\cdots x_n^{(u_n)} f(\pi_1,\dots,\pi_n) x_1^{(-u_1)}\cdots x_n^{(-u_n)} = f(\pi_1-u_1,\dots,\pi_n-u_n)\pi_1^{(u_1)}\cdots\pi_n^{(u_n)}$$

This leads to a tractable criterion for the simplicity of D^λ.

D^λ is simple if and only if for all $f \in D$, not divisible by $\pi-\lambda$, the polynomials in $\pi_1,\dots,\pi_n$

$$\pi - \lambda = \sum a_i \pi_i - \lambda$$

and all

$$f(\pi_1-u_1,\dots,\pi_n-u_n)\pi_1^{(u_1)}\cdots\pi_n^{(u_n)}$$

where $(u_i)_i \in \mathbb{Z}^n$, $\sum a_i u_i = 0$, have no common zero.

We will now investigate this condition for a particular f. For $H \in D$, let $Z(H)$ denote the zeroes of H. Then the above condition may be rephrased as follows :

D^λ is simple if and only if for all $f \in D$, not divisible by $\pi-\lambda$ and for all $(\alpha_i)_i \in k^n$, $\sum a_i\alpha_i = \lambda$ there exist $(u_i)_i \in \mathbb{Z}^n$, $\sum a_i u_i = 0$ such that

$$(\alpha_1-u_1,\dots,\alpha_n-u_n) \notin Z(f)$$

and

$$\text{for all } i \quad \begin{array}{lcl} u_i \geq 0 & \Rightarrow & \alpha_i \notin [0,u_i-1]\cap\mathbb{Z} \\ u_i < 0 & \Rightarrow & \alpha_i \notin [u_i,-1]\cap\mathbb{Z} \end{array} \tag{12}$$

(12) may be restated in a more convenient way as :

$$\text{for all } i \text{ either } \alpha_i \notin \mathbb{Z} \quad\text{or}\quad \begin{array}{lcl} \alpha_i \geq 0 & \Rightarrow & u_i \leq \alpha_i \\ \alpha_i < 0 & \Rightarrow & u_i > \alpha_i \end{array} \tag{13}$$

Fix a particular set $(\alpha_i)_i \in k^n$ such that $\sum a_i\alpha_i = \lambda$

Assume first that some $\alpha_j \notin \mathbb{Z}$. If $a_j = 0$ then we may, without changing λ, replace α_j by 0, and then condition (13) becomes harder to satisfy.

Therefore, we must only consider the case $a_j \neq 0$. Then the set of $(u_1, \dots, u_n)$ satisfying (13) contains

$$\begin{aligned} U \quad = \quad & \{(u_1, \dots, u_n) \in \mathbb{Z}^n \mid \sum_i a_i u_i = 0 \text{ and } \forall i \neq j \\ & \alpha_i \geq 0 \Rightarrow u_i \leq \alpha_i \\ & \alpha_i < 0 \Rightarrow u_i > \alpha_i\} \end{aligned}$$

and it follows from lemma 3.4 that U is Zarisky dense in $Z(\pi)$. To see this we have to find $(u_i)_i \in \mathbb{Q}^n$, $\sum a_i u_i = 0$ such that for all $i \neq j$

$$\begin{aligned} \alpha_i \geq 0 \quad &\Rightarrow \quad u_i < 0 \\ \alpha_i < 0 \quad &\Rightarrow \quad u_i > 0 \end{aligned}$$

The existence of such $(u_i)_i$ follows from the fact that $a_i \neq 0$.

From this one concludes that it is possible to choose $(u_1, \dots, u_n) \in U$ such that $(\alpha_1 - u_1, \dots, \alpha_n - u_n) \in Z(\pi - \lambda) \setminus Z(f)$ since $Z(f) \cap Z(\pi - \lambda) \neq Z(\pi - \lambda)$. Hence if $\lambda \in k \setminus \mathbb{Z}\gcd(a_i)_i$ there will always exist $\alpha_j \notin \mathbb{Z}$ and hence D^λ is simple.

So we may restrict ourselves to $\lambda \in \mathbb{Z}\gcd(a_i)$ and $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$.

Assume that one of the following conditions holds

(1) $\exists \alpha_i, \alpha_j \geq 0, a_i a_j < 0$
(2) $\exists \alpha_i, \alpha_j < 0, a_i a_j < 0$
(3) $\exists \alpha_i \geq 0, \alpha_j < 0, a_i a_j > 0$

Now let U' be the set of $(u_1, \dots, u_n) \in \mathbb{Z}^n$ such that (13) holds. Then it follows again from lemma 3.4 that U' is Zarisky dense in $Z(\pi)$ which implies that we may choose $(u_1, \dots, u_n) \in U'$ such that $(\alpha_1 - u_1, \dots, \alpha_n - u_n) \in Z(\pi - \lambda) \setminus Z(f)$.

It is easily verified that the set of those λ such that for all $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n : \sum a_i \alpha_i = \lambda$, condition 1.,2. or 3. holds is exactly $\mathcal{B}$. Hence if $\lambda \in \mathcal{B}$ then D^λ is simple.

Now let $\lambda \in \mathcal{A}$ and choose $(\alpha_i)_i \in \mathbb{Z}^n$ such that $\sum a_i \alpha_i = \lambda$ and

$$\begin{aligned} a_i \leq 0 \quad &\Rightarrow \quad \alpha_i < 0 \\ a_i > 0 \quad &\Rightarrow \quad \alpha_i \geq 0 \end{aligned}$$

Then it follows from lemma 3.4 and condition 2.1 that U' is *not* Zarisky dense in $Z(\pi)$. Hence we may find $f \in D$, zero on all $(\alpha_1 - u_1, \dots, \alpha_n - u_n)$ for $(u_1, \dots, u_n) \in U'$ but not zero on the whole of $Z(\pi - \lambda)$. Such an f will generate a non-trivial ideal in D^λ.

The case $\lambda \in \mathcal{C}$ is treated similarly.

Hence we have proved the following theorem

Theorem 3.1. *D^λ is simple if and only if $\lambda \in \mathcal{B} \cup \mathcal{E}$.*

Remark 3.2. It is possible to give a different proof of this result with the following method.

Let $S = k[y_1, \ldots, y_n]$ and embed R in S via $x_i^{|a_i|} = y_i$ and let $G = \mathbb{Z}/(a_1) \times \cdots \times \mathbb{Z}/(a_n)$. Then $R = S^G$ and hence $D(S)^G \hookrightarrow D(R)$ (not equality since S/R is ramified in codimension one). Also $\pi = \sum a_i x_i \partial_i = \sum \epsilon y_i \delta_i$ where $\delta_i = \frac{\partial}{\partial y_i}$, $\epsilon_i = \frac{a_i}{|a_i|}$.

Then one can prove that the natural map

$$(D(S)_0/D(S)_0(\pi - \lambda))^G = D(S)_0^G/D(S)_0^G(\pi - \lambda) \to D(R)_0/D(R)_0(\pi - \lambda)$$

is an inclusion. Furthermore, both rings are domains, contained in the same field of fractions. Hence if $D(S)_0/D(S)_0(\pi - \lambda)$ is simple then D^λ is also simple (using [10, prop. 7.8.12]).

Hence we may reduce to the case $a_i = -1, 0, 1$. The a_i's that are zero can be dispensed with easily, and hence we may assume that $a_i = \pm 1$ and then using cor. 2.3 we arrive at the case where all $a_i = 1$.

Then D^λ is a ring of twisted differential operators on projective n-space (see Section 6). Here everything is well known, at least when $\lambda \in \mathbb{Z}$ [3]. Presumably the case $\lambda \notin \mathbb{Z}$ is also well understood, but I have been unable to locate a precise reference.

Now to prove a complete statement as Theorem 3.1, it turns out that one also has to throw in the Morita equivalences among the D^λ's constructed in Section 4.

In this way the proof becomes rather involved, and unelegant. Therefore, we decided to include a direct proof; a proof which has the advantage that it may be generalized to higher dimensional tori.

Remark 3.3. In Theorem 3.1 the non-simplicity of D^λ for $\lambda \in \mathcal{A} \cup \mathcal{C}$ may also be proved directly by constructing explicit finite dimensional representations. E.g. if all $a_i > 0$ and if $\lambda \in \mathcal{A}$ then $R/(\pi - \lambda)R = R_\lambda$ is a finite dimensional D^λ-module.

The following elementary lemma has been used repeatedly

Lemma 3.4. *Assume that E is a $\mathbb{Q}$-vector space, and L is a lattice spanning E. Let $\lambda_1, \ldots, \lambda_t \in E^*$, $c_1, \ldots, c_j, c_{j+1}, \cdots, c_t \in \mathbb{Q}$ and define*

$$C = \{x \in E \mid \langle \lambda_i, x\rangle \le c_i \text{ for } i \in \{1, \ldots j\} \\ \langle \lambda_i, x\rangle < c_i \text{ for } i \in \{j+1, \ldots t\}\}$$

Then $C \cap L$ will be Zarisky dense in E if and only if there exist $y \in E$ such that $\langle \lambda_i, y\rangle < 0$ for all i.

The proof will be based on two sublemmas.

Lemma 3.5. *Let $c \in \mathbb{Q}$, $\lambda \in E^*$. Then there exist $c' \in \mathbb{Q}$ such that*

$$\{l \in L \mid \langle \lambda, l \rangle < c\} = \{l \in L \mid \langle \lambda, l \rangle \leq c'\}$$

Proof. We may assume that $E = \mathbb{Q}$, $L = \mathbb{Z}^n$, $\lambda(x) = \sum v_i x_i$ where $v_i = p_i/q_i \in \mathbb{Q}$. Then $\lambda(x)$ is always a multiple of $\frac{1}{q_1 \cdots q_n}$. Hence

$$\{\langle \lambda, l \rangle \mid l \in L, \langle \lambda, l \rangle < c\}$$

has a maximum which plays the role of c'. □

Lemma 3.6. *Let c, $c' \in \mathbb{Q}$, $\lambda \in E^*$. Then the set*

$$\{l \in L \mid c \leq \langle \lambda, l \rangle \leq c'\} \tag{14}$$

is not Zarisky dense in E.

Proof. As in the proof of the previous lemma $\langle \lambda, l \rangle$ must be a multiple of some fixed element of $\mathbb{Q}$. Hence (14) is contained in a finite union of hyperplanes. □

Proof. **of lemma 3.4** We may assume that $E = \mathbb{Q}^n$ and $L = \mathbb{Z}^n$. Additionally, using lemma 3.5 and 3.6 we may assume that C has the form

$$C = \{x \in E \mid \forall i \, \langle \lambda_i, x \rangle < 0\}$$

Then the $\Rightarrow$-direction is trivial. Hence we concentrate on the $\Leftarrow$-direction. C contains a small ball around y and hence C is Zarisky dense in E. It therefore suffices to show that $C \cap L$ is Zarisky dense in C. Let $z \in C$. Then there exist $v \in \mathbb{N} \setminus \{0\}$ such that $vz \in L$. Hence all positive multiples of vz lie in $C \cap L$. But then $\{tz \mid t \in \mathbb{Q}^+\}$ is in the Zarisky closure of $C \cap L$ in C and therefore z itself is in this Zarisky closure. □

4. Morita equivalence of the D^λ's

In this section we will use the same notations as in the previous sections.

Additionally if $l \in \mathbb{Z}$ and $\lambda' = \lambda + l$ then we will write $\lambda \to \lambda'$ if in the Morita context (3)

$$(D^\lambda, D^{\lambda'}, D^\lambda(-l), D^{\lambda'}(l), i, j) \tag{15}$$

j is surjective. I.e. if the following condition

$$(\pi - \lambda')D(R)_0 + D(R)_l D(R)_{-l} = D(R)_0 \tag{16}$$

is satisfied.

It is clear that $\lambda \to \lambda'$ is transitive. Furthermore if $\lambda \to \lambda'$ and $\lambda' \to \lambda$ then the Morita context (15) is a Morita equivalence, and hence D^λ and $D^{\lambda'}$ are Morita equivalent.

Finally, general yoga about Morita contexts yields the following result [10, Prop. 3.4.5].

Proposition 4.1. *If $\lambda \to \lambda'$ then $D^{\lambda'}(l)$ is right D^λ-projective and $D^\lambda(-l)$ is left D^λ-projective.*

Using the $\mathbb{Z}^n$-grading on $D(R)$ and (10), (16) is equivalent with

$$(\pi - \lambda')D + \sum_{\substack{u_1,\dots,u_n \in \mathbb{Z} \\ \sum a_i u_i = l}} x_1^{(u_1)} \cdots x_n^{(u_n)} D x_1^{(-u_1)} \cdots x_n^{(-u_n)} = D$$

which is equivalent with

$$(\pi - \lambda')D + \sum_{\substack{u_1,\dots,u_n \in \mathbb{Z} \\ \sum a_i u_i = l}} \pi_1^{(u_1)} \cdots \pi_n^{(u_n)} D = D$$

I.e. there will be an arrow $\lambda \to \lambda'$ if the polynomials in D

$$\sum a_i \pi_i - \lambda'$$

$$\pi_1^{(u_1)} \cdots \pi_n^{(u_n)}, \quad \sum a_i u_i = l$$

have no common zeroes.

This may be rephrased as follows :

There is an arrow $\lambda \to \lambda'$ if for all $(\alpha_i)_i \in k^n$, $\sum a_i \alpha_i = \lambda'$ there exist $(u_i)_i \in \mathbb{Z}^n$, $\sum a_i u_i = l$ such that

$$\text{for all } i \quad \begin{array}{lcl} u_i \geq 0 & \Rightarrow & \alpha_i \notin [0, u_i - 1] \cap \mathbb{Z} \\ u_i < 0 & \Rightarrow & \alpha_i \notin [u_i, -1] \cap \mathbb{Z} \end{array} \tag{17}$$

(Note the similarity with (12).)

(17) may be rewritten as

$$\text{(18)} \qquad \text{for all } i \text{ either } \alpha_i \notin \mathbb{Z} \quad \text{or} \quad \begin{array}{ccc} \alpha_i \geq 0 & \Rightarrow & u_i \leq \alpha_i \\ \alpha_i < 0 & \Rightarrow & u_i > \alpha_i \end{array}$$

Assume first that some $\alpha_j \notin \mathbb{Z}$. If $a_j = 0$ then we may, without changing λ, replace α_j by 0 and then (18) becomes harder to satisfy. Therefore we must only consider the case $a_j \neq 0$ but then it is easy to see that one can satisfy (18) if and only if $\gcd(a_i)_i \mid l$.

Hence the relation $\lambda \to \lambda'$ induces an equivalence relation on $\mathcal{E}$ with equivalence classes $\mathcal{E}_\mu$, $\mu \in k$. Furthermore there are no arrows going in and out of $\mathcal{E}$.

Hence we may now restrict ourselves to λ, $\lambda' \in \mathbb{Z}\gcd(a_i)_i$ and $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. By defining $\beta_i = \alpha_i - u_i$, we obtain the following very symmetric statement.

There is an arrow $\lambda \to \lambda'$ if for all $(\alpha_i)_i \in \mathbb{Z}^n$, $\sum a_i\alpha_i = \lambda'$ there exist $(\beta_i)_i \in \mathbb{Z}^n$, $\sum a_i\beta_i = \lambda$ such that

$$\text{(19)} \qquad \text{for all } i \quad \begin{array}{ccc} \alpha_i \geq 0 & \Rightarrow & \beta_i \geq 0 \\ \alpha_i < 0 & \Rightarrow & \beta_i < 0 \end{array}$$

If $\lambda' \in \mathcal{B}$ then one of the conditions 1.,2.,3. listed in the proof of Theorem 3.1 must hold and then we can always find $(\beta_i)_i \in \mathbb{Z}^n$ satisfying (19).

If $\lambda' \in \mathcal{A}$ then it follows from (19) and the definition of $\mathcal{A}$ that also $\lambda \in \mathcal{A}$.

Similarly $\lambda' \in \mathcal{C}$ implies $\lambda \in \mathcal{C}$. We have now proved

Proposition 4.2. *All elements of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}_\mu$, $\mu \in k$ are equivalent under the relation $\lambda \to \lambda'$. The only additional arrows are those going from $\mathcal{A}$ to $\mathcal{B}$ and from $\mathcal{C}$ to $\mathcal{B}$.*

Corollary 4.3. *For all $\lambda \in \mathcal{A} \cup \mathcal{C}$, the rings D_λ are mutually Morita equivalent. The same is true for all $\lambda \in \mathcal{B}$ and for all $\lambda \in \mathcal{E}_\mu \cup \mathcal{E}_{-\mu}$, $\mu \in k$.*

Furthermore, if $\lambda \in \mathcal{B}$ and $\lambda' \notin \mathcal{B}$ then D^λ and $D^{\lambda'}$ are not *Morita equivalent.*

Proof. The first part follows from 4.2 and 2.3. For the last part we would like to invoke 5.5. However, we can't do this directly since 5.5 depends on 4.5, which in turn depends on this corollary. However it is easily seen that 4.5 depends only on the case $\lambda \in \mathcal{B}$ and $\lambda' \in \mathcal{A} \cup \mathcal{C}$ which follows from 3.1. This breaks the circle. □

Remark 4.4. If λ, $\lambda' \in \mathcal{E}$ and D^λ is Morita equivalent with $D^{\lambda'}$ then we don't know whether necessarily λ, $\lambda' \in \mathcal{E}_\mu \cup \mathcal{E}_{-\mu}$ for some $\mu \in k$

The following partial converse to prop. 4.1 will be needed in the next section.

Lemma 4.5. *If $\lambda \in \mathcal{B}$ and $\lambda' \in \mathcal{A} \cup \mathcal{C}$ then $D^\lambda(-l)$ is not a projective left D^λ-module.*

Proof. Using prop. 4.1 and prop. 4.2 it follows that $D^\lambda(-l)$ is right $D^{\lambda'}$-projective. Then the lemma below together with 2.3 implies the result. □

Lemma 4.6. *Suppose that A and B are maximal orders in the same field of fractions. Suppose furthermore that P is an A-B fractional ideal, projective on either side. Then A and B are Morita equivalent.*

Proof. It is sufficient to show that the canonical maps

$$A \to \mathrm{End}_B(P)$$

$$B^\circ \to \mathrm{End}_A(P)$$

are isomorphisms.

Now $\mathrm{End}_B(P)$ is an order in the fraction field of A, which contains A and which is equivalent to A. Hence $A = \mathrm{End}_B(P)$. The proof that $B^\circ = \mathrm{End}_A(P)$ is similar. □

5. The global dimension of D^λ

In this section, the notations and conventions of the previous sections will remain in force.

We start with the case were all $a_i \geq 0$, and in particular : $a_1, \dots, a_l > 0$, $a_{l+1}, \dots, a_n = 0$. Note that condition 2.1 implies that $l > 0$.

First let $\lambda \in \mathcal{A} \cup \mathcal{E}$, and f a homogeneous element in R.

Lemma 5.1. *Under the current assumptions, D^λ_f is a right flat D^λ-module.*

Proof. We may assume that f has strictly positive degree, otherwise D^λ_f is an Öre localization of D^λ and then the result is clear.

Let $\deg f = u > 0$. Then

$$D^\lambda_f = \operatorname{inj}\lim_t D^{\lambda+tu}(tu) \tag{20}$$

as right D^λ-modules. Now $u \mid \gcd(a_i)_i$, and then from the definitions of $\mathcal{A}$ and $\mathcal{E}$ it is clear that for any t_0 there is a $t \geq t_0$ such that $\lambda + tu \in \mathcal{A} \cup \mathcal{E}$. Hence by prop. 4.2 and 4.1, D^λ_f is right flat. □

Now let K be the complex of $D(R)$-bimodules

$$0 \to D(R) \to \bigoplus_{i \in \{1,\dots,l\}} D(R)_{x_i} \to \bigoplus_{\substack{i,j \in \{1,\dots,l\} \\ i \neq j}} D(R)_{x_i x_j} \to \cdots \xrightarrow{d} D(R)_{x_1 \cdots x_l} \to 0$$

with the standard alternating boundary maps.

By going to the associated graded modules, we see that K is everywhere exact, except in $D(R)_{x_1 \cdots x_l}$ and

$$\operatorname{gr}(\operatorname{coker} d) = x_1^{-1} \cdots x_l^{-1} k[x_1^{-1}, \dots, x_l^{-1}, x_{l+1}, \cdots, x_n, \partial_1, \dots, \partial_n]$$

Hence K becomes exact everywhere when restricted to degree zero. (Here we use the fact that $l > 0$).

In the resulting complex the modules have no $\pi - \lambda$ torsion, and hence after tensoring with D^λ, we obtain an exact sequence K^λ of right flat D^λ bimodules.

$$0 \to D^\lambda \to \bigoplus_{i \in \{1,\dots,l\}} D^\lambda_{x_i} \to \bigoplus_{\substack{i,j \in \{1,\dots,l\} \\ i \neq j}} D^\lambda_{x_i x_j} \to \cdots \to D^\lambda_{x_1 \cdots x_l} \to 0$$

Lemma 5.2. *Let $\{i_1, \dots, i_p\} \subset \{1, \dots, l\}$. Then $D^\lambda_{x_{i_1} \cdots x_{i_p}}$ has finite global dimension.*

Proof. It is easy to see that $D^\lambda_{x_{i_1} \cdots x_{i_p}}$ is an Öre localization of $D^\lambda_{x_{i_1}}$. Hence it is sufficient to show that $D^\lambda_{x_i}$ has finite global dimension for $i \in \{1, \dots, l\}$.

Let $R' = k[x_1, \ldots, \hat{x}_i, \ldots, x_n]$. Then it is easy to see that the map

$$\bigoplus_t D(R')_{ta_i} \to D^\lambda_{x_i} : r \mapsto \overline{x_i^{-t} r}$$

is an isomorphism. Hence

$$D^\lambda_{x_i} \cong D(R')^{\mathbb{Z}/a_i\mathbb{Z}}$$

which implies that $D^\lambda_{x_i}$ has finite global dimension [10, prop. 7.8.11]. □

Now we need the following elementary lemma.

Lemma 5.3. *Let B/A be a right flat ring extension and assume that I is a left injective B-module. Then the restriction of I to A is left injective.*

Proof. For every left ideal J in A, and every A-linear map $\phi : J \to I$, we have to find a factorization through A. Now ϕ extends to a map $\phi' : B \otimes_A J \to I$, and from the flatness of B it follows that $B \otimes_A J$ is a left ideal in B. Hence ϕ' factors through B, but this immediately provides us with a factorization of ϕ through A. □

Now assume that M is a left D^λ-module. Tensoring with D^λ gives an exact sequence

$$K^\lambda \otimes_{D^\lambda} M$$

Now in this sequence, each of the modules $D^\lambda_{x_{i_1}\cdots x_{i_p}} \otimes M$ has a finite injective resolution as $D^\lambda_{x_{i_1}\cdots x_{i_p}}$-module (lemma 5.2). However, by lemma 5.3, these resolutions are also injective resolutions as D^λ-modules. Hence M has finite injective dimension.

We conclude that if $\lambda \in \mathcal{A} \cup \mathcal{E}$ then D^λ has finite global dimension.

This is the most suitable place to insert a lemma that will be used in the next section.

Lemma 5.4. *If $\lambda \in \mathcal{A} \cup \mathcal{E}$ then $U = \oplus D^\lambda_{x_i}$ is a right faithfully flat D^λ-module.*

Proof. Let M be a left D^λ-module. Then the complex

$$K^\lambda \otimes_{D^\lambda} M$$

is exact. In particular $U \otimes_{D^\lambda} M = 0$ implies $M = 0$. □

Assume now that $\lambda \in \mathcal{C}$. Then we may use cor. 2.3, and the fact that $-\lambda - \sum a_i \in \mathcal{A}$ to show that in this case D^λ also has finite global dimension.

Finally, suppose that $\lambda \in \mathcal{B}$. We will show that now D^λ has infinite global dimension.

First look at the Koszul complex (of left modules) associated to the regular sequence $x_1, \ldots, x_l$.

$$0 \to D(R)(-a_1-\cdots-a_l) \to \cdots \to \bigoplus_{\substack{i,j\in\{1,\ldots,l\}\\ i\neq j}} D(R)(-a_i-a_j) \to \bigoplus_{i\in\{1,\ldots,l\}} D(R)(-a_i) \xrightarrow{d} D(R)$$

This complex is exact, and

$$\operatorname{gr}(\operatorname{coker} d) = k[x_{l+1}, \ldots, x_n, \partial_1, \ldots, \partial_n]$$

Hence if

$$u \notin -\mathbb{N}a_1 - \cdots - \mathbb{N}a_n \tag{21}$$

there is an exact sequence

$$0 \to D(R)_{u-a_1-\cdots-a_l} \to \cdots \to \bigoplus_i D(R)_{u-a_i} \to D(R)_u \to 0$$

and since there is no $\pi - \lambda$ torsion, tensoring with D^λ yields an exact sequence

$$0 \to D^\lambda(u-a_1-\cdots-a_l) \to \cdots \to \bigoplus_i D^\lambda(u-a_i) \to D^\lambda(u) \to 0 \tag{22}$$

Now look at the Koszul complex associated to the regular sequence $\partial_1, \ldots, \partial_l$.

$$0 \to D(R)(a_1+\cdots+a_l) \to \cdots \to \bigoplus_{\substack{i,j\in\{1,\ldots,l\}\\ i\neq j}} D(R)(a_i+a_j) \to \bigoplus_{i\in\{1,\ldots,l\}} D(R)(a_i) \xrightarrow{d} D(R)$$

This time

$$\operatorname{gr}(\operatorname{coker} d) = k[x_1, \ldots, x_n, \partial_{l+1}, \ldots, \partial_n]$$

Assume now that

$$u - a_1 - \cdots - a_l \notin \mathbb{N}a_1 + \cdots + \mathbb{N}a_n \tag{23}$$

then restricting to degree $u - a_1 - \cdots - a_n$ and tensoring with D^λ yields an exact sequence

(24)

$$0 \to D^\lambda(u) \to \cdots \to \bigoplus_i D^\lambda(u-a_1-\cdots-\hat{a}_i-\cdots-a_n) \to D^\lambda(u-a_1-\cdots-a_n) \to 0$$

Our aim is now to select u in such a way that

$$\lambda - u \notin \mathcal{B} \qquad \lambda - u + \sum_{i\in\{1,\ldots,l\}} a_i \notin \mathcal{B}$$

$$\forall I \subsetneq \{1, \ldots, l\}, I \neq \emptyset : \lambda - u + \sum_{i\in I} a_i \in \mathcal{B}$$

Then according to lemma 4.5 $D^\lambda(u)$ and $D^\lambda(u-a_1-\cdots-a_l)$ will not be projective, whereas all the other modules, occurring in the complexes (22) and (24) are projective. This clearly implies that D^λ has infinite global dimension.

From lemma 2.5 it follows that the minimal element of $\mathcal{A}$ and the maximal element of $\mathcal{C}$ are respectively 0 and $-\sum a_i$. Hence it suffices to put

$$\lambda - u = -\sum_{i\in\{1,\ldots,l\}} a_i$$

or

$$u = \lambda + \sum_{i\in\{1,\ldots,l\}} a_i$$

We still have to verify that (21) and (23) hold. Now if (21) does not hold then $\lambda \in \mathcal{C}$. Similarly if (23) does not hold then $\lambda \in \mathcal{A}$. This yields a contradiction since $\lambda \in \mathcal{B}$.

Now we let the a_i's be arbitrary. Put $b_i = |a_i|$ and $\lambda' = \lambda - \sum_{a_i<0} |a_i|$. Let $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$, $\mathcal{E}'$, $D'^{\lambda'}$, π' be defined as in cor. 2.3 and lemma 2.6. Then by what we have shown above D^λ will have finite global dimension if and only if $D'^{\lambda'}$ has finite global dimension if and only if $\lambda' \notin \mathcal{B}'$ if and only if $\lambda \notin \mathcal{B}$.

Hence we have now proved :

Theorem 5.5. *D^λ has finite global dimension if and only if $\lambda \notin \mathcal{B}$.*

6. Applications

6.1. **Weighted projective spaces.** In this section we will assume that all $a_i > 0$.

Put $Y = \operatorname{Proj} R$. In addition to condition 2.1 we will assume the following condition

Condition 6.1.1.

$$\forall j \in \{1, \dots, n\}, \quad \gcd(a_i)_{i \neq j} = 1$$

We may assume that this condition holds, without changing Y.

For a graded R-module, let $\tilde{M}$ be the corresponding quasi-coherent sheaf on Y. For $l \in \mathbb{Z}$ let $\mathcal{O}_Y(l) = \widetilde{R(l)}$.

For a quasi coherent $\mathcal{O}_Y$-module, denote by $\mathcal{D}_Y(\mathcal{F})$ its sheaf of differential operators and $D_Y(\mathcal{F}) = \Gamma(Y, \mathcal{D}(\mathcal{F}))$ its ring of differential operators.

We also define

$$\mathcal{D}^\lambda = \widetilde{D(R)}/\widetilde{D(R)}(\pi - \lambda)$$

Then we may prove the following theorem.

Theorem 6.1.2. *$D^\lambda = \Gamma(Y, \mathcal{D}^\lambda)$ and for $l \in \mathbb{Z}$: $\mathcal{D}^l = \mathcal{D}_Y(\mathcal{O}_Y(l))$.*

Proof. To prove $D^\lambda = \Gamma(Y, \mathcal{D}^\lambda)$ it is sufficient to prove that $H^1(Y, \widetilde{D(R)}) = 0$.

If we filter $D(R)$ by order of differential operators we obtain

$$\operatorname{gr} D(R) = R[\partial_1, \dots, \partial_n] = \bigoplus_{(u_i)_i \in \mathbb{N}^n} R(u_1 a_1 + \dots + u_n a_n)$$

Hence a filtration is induced on $\widetilde{D(R)}$ with associated graded quotients

$$\bigoplus_{(u_i)_i \in \mathbb{N}^n} \mathcal{O}_Y(u_1 a_1 + \dots + u_n a_n)$$

Therefore, by an obvious generalization of Serres computation of cohomology on projective spaces, $H^1(Y, \widetilde{D(R)}) = 0$.

To prove that the canonical map $\mathcal{D}^l \to \mathcal{D}(\mathcal{O}_Y(l))$ is an isomorphism, we may restrict to the basic open sets $Y_i = \{x_i \neq 0\}$. By definition

$$\Gamma(Y_i, \mathcal{D}^l) = D^l_{x_i}, \quad \Gamma(Y_i, \mathcal{D}_Y(\mathcal{O}_Y(l))) = D_{(R_{x_i})_0}((R_{x_i})_l)$$

Now, by a straightforward generalization of [12, Th. 7.11] to covariants, we find that the canonical map

$$D^l_{x_i} \to D((R_{x_i})_l)$$

is an isomorphism (uses condition 6.1.1). □

Hence Theorems 3.1, 5.5 and Corollary 4.3 may be applied to twisted differential operators of weighted projective spaces.

We include one more result, which is a generalization of the famous Bernstein-Beilinson theorem to weighted projective spaces.

Theorem 6.1.3. *If $\lambda \in \mathcal{A} \cup \mathcal{E}$ then the functors $\Gamma(Y, -)$, $\mathcal{D}^\lambda \otimes_{D^\lambda} -$ define mutually inverse equivalences between the category of quasi-coherent sheaves of left $\mathcal{D}^\lambda$-modules and the category of left D^λ-modules.*

Proof. This follows from [6] and lemma 5.4. □

6.2. **Covariants for a one-dimensional torus.** In this section T will be a one-dimensional torus. We will identify the character group $\mathrm{Hom}(T, \mathbb{G}_m)$ with $\mathbb{Z}$ where 1 corresponds to the identity.

If A is a k-algebra on which T acts rationally then we denote by A_l the part of degree A with weight l. I.e.

$$A_k = \{a \in A \mid \forall z \in T : z.a = z^l a\}$$

Clearly $A_0 = A^T$. Furthermore $A = \oplus A_l$ is a grading on A.

Conversely, if we are given a grading on A then it may be converted into a T-action. Hence we will consider the T-action on R corresponding to the grading on R.

Without changing R^T, we may assume that the condition 6.1.1 holds, which is what we will do.

Furthermore we will put on the slightly restrictive condition :

Condition 6.2.1. *There are at least two strictly positive and at least two strictly negative a's*

Then there is the following result.

Theorem 6.2.2. *Assume that $l \in \mathbb{Z}$. Then $D^l = D_{R^T}(R_l)$.*

Proof. This follows directly form a generalization of [12, Th. 7.11] to covariants (uses condition 6.1.1 and 6.2.1). □

Hence we may apply Theorems 3.1, 5.5 and Corollary 4.3 to rings of differential operators of modules of covariants.

Using [13, Th. 3.3 and cor. 3.4] or [15, 4.1] we can state the following somewhat curious corollary.

Corollary 6.2.3. *Let $l \in \mathbb{Z}$. Then R_l is a Cohen-Macaulay R^T-module if and only if $D_{R^T}(R_l)$ is simple, if and only if $D_{R^T}(R_l)$ does not have finite global dimension.*

Or more specifically.

Corollary 6.2.4. *$D(R^T)$ is simple and has infinite global dimension.*

It was the following partial generalization of cor. 6.2.3 that has originally prompted me to look into these matters.

Theorem 6.2.5. *Let A be a finitely generated commutative k-algebra, and M a finitely generated A-module. Then if $D_A(M)$ is simple then M is Cohen-Macaulay.*

Proof. Assume that $D_A(M)$ is simple. Let $X = \operatorname{Spec} A$ and let $\mathcal{M}$ be the quasi-coherent $\mathcal{O}_X$-module associated to M. Let Z be the support of $\mathcal{M}$ and let Y be the non-Cohen-Macaulay locus of M. Put $u = \operatorname{codim}(Y, Z)$. Then according to [4] $U_i = H^i_Y(\mathcal{M})$ is coherent for $0 \leq i < u$. But then $\operatorname{Ann}_{D_A(M)} U_i$ is non-zero, and hence equal to $D_A(M)$.

This implies that $U_i = 0$ for $0 \leq i < u$. Hence if η is a generic point of Y of codimension u in Z, then $\mathcal{M}_\eta$ is Cohen-Macaulay. This contradicts the fact that Y was the non-Cohen-Macaulay locus of $\mathcal{M}$. □

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