INVARIANTS UNDER TORI OF RINGS OF DIFFERENTIAL OPERATORS AND RELATED TOPICS

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Abstract. If G is a reductive algebraic group acting rationally on a smooth affine variety X then it is generally believed that $D(X)^G$ has properties very similar to those of enveloping algebras of semisimple Lie algebras. In this paper we show that this is indeed the case when G is a torus and $X = k^r \times (k^*)^s$. We give a precise description of the primitive ideals in $D(X)^G$ and we study in detail the ring theoretical and homological properties of the minimal primitive quotients of $D(X)^G$. The latter are of the form $D(X)^G/(\mathfrak{g} - \chi(\mathfrak{g}))$ where $\mathfrak{g} = \mathrm{Lie}(G)$, $\chi \in \mathfrak{g}^*$ and $\mathfrak{g} - \chi(\mathfrak{g})$ is the set of all $v - \chi(v)$ with $v \in \mathfrak{g}$. They occur as rings of twisted differential operators on toric varieties.

As a side result we prove that if G is a torus acting rationally on a smooth affine variety then $D(X/\!\!/G)$ is a simple ring.

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1. Introduction

Throughout this paper k will be an algebraically closed base field of characteristic zero. Let G be a connected reductive group acting on a smooth affine variety X. Of considerable interest is the ring of invariant differential operators $D(X)^G$. For example if $X = G/K$ is a symmetric space then Harish-Chandra studied $D(X)^G$ in order to gain insight into various function spaces attached to X.

Recently Knop [10] succeeded in generalizing one of the most fundamental results of Harish-Chandra. That is, he was able to give a precise description of the center of $D(X)^G$. In particular he shows that it is always a polynomial ring. If one considers the action of $G \times G \to G$ then this yields the Harish-Chandra isomorphism for $U(\mathfrak{g})$, $\mathfrak{g} = \mathrm{Lie}(G)$. So this result by Knop, together with explicit computations in specific cases, suggests that $D(X)^G$ should have properties very similar to those of enveloping algebras. In this paper we show that this is the case when G is a torus.

There are other reasons for studying $D(X)^G$. If Y is a non-smooth variety then usually $D(Y)$ is very badly behaved [4]. However when $Y = X/\!\!/G$ it is a general feeling that $D(Y)$ should have various nice properties. More precisely, one can make the following conjecture

Conjecture 1.1. (f) $D(X/\!\!/G)$ is finitely generated and noetherian.

(s) $D(X/\!\!/G)$ is a simple ring.

The restriction homomorphism $D(X)^G \to D(X/\!\!/G)$ defines an algebra map which in many cases is surjective and has kernel $(D(X)\mathfrak{g})^G$ [18][22]. If this is true then conjecture 1.1(f) follows trivially. In this way one can prove 1.1(f) for tori and for classical representations of classical groups [14][18]. More generally 1.1(f) is true whenever W is "big enough" in an appropriate, but somewhat technical sense.

In contrast with 1.1(f), not much is known about 1.1(s), see [14][28][31]. Nevertheless conjecture 1.1(s) is important since it implies the Hochster-Roberts Theorem [28]. It is clear that a detailed understanding of $D(X)^G$ might be instrumental in proving this conjecture in general.

There is an obvious generalization to covariants. If χ is an irreducible character of G then we denote by $\mathcal{O}_{X,\chi}$ the isotypical component of $\mathcal{O}_X$ associated to χ, considered as a coherent sheaf of $\mathcal{O}_{X/\!\!/G}$-modules. Then there is again a restriction map $D(X)^G \to D(\mathcal{O}_{X,\chi})$ which is surjective in most cases [22]. Since the simplicity of $D(\mathcal{O}_{X,\chi})$ (or even of the image of $D(X)^G$) implies that $\mathcal{O}_{X,\chi}$ is Cohen-Macaulay, an understanding of $D(X)^G$ may shed new light upon the Cohen-Macaulayness problem for modules of covariants [24][30].

In the current paper we prove conjecture 1.1(s) for tori. That is, we prove (see §7.5) :

Theorem A. *Assume that G is a torus, acting rationally on a smooth affine variety X. Then $D(X/\!\!/G)$ is simple.*

When G is a torus there is yet another motivation for studying $D(X)^G$. If Y is a toric variety then it is shown in [19] that there exist r, s such that for $X = k^r \times (k^*)^s$ and for any invertible sheaf $\mathcal{L}$ on Y there is a surjective map

$$D(X)^G \to D(\mathcal{L})$$

whose kernel is generated by

$$\mathfrak{g} - \chi(\mathfrak{g}) \stackrel{\text{def}}{=} \{v - \chi(v) \mid v \in \mathfrak{g}\}$$

where χ is a character of G, depending on $\mathcal{L}$.

Let us now summarize the other results in this paper. With the case of a toric variety in mind, these are stated for $X = k^r \times (k^*)^s$. This is an important special case, and the Luna Slice Theorem makes it often possible to reduce the general case to it. This is in fact the strategy we follow to prove Theorem A above.

If $X = k^r \times (k^*)^s$, $n = r + s$ then

$$D(X) = k[x_1, \dots, x_r, x_{r+1}^{\pm 1}, \dots, x_n^{\pm 1}, \partial_1, \dots, \partial_n] \tag{1.1}$$

where $\partial_i = \frac{\partial}{\partial x_i}$. In the rest of this introduction, and also in most of the paper, we denote this ring simply by A.

One easily shows that the center of A^G is given by (the image of) the symmetric algebra of $\mathfrak{g}$. Hence every character $\chi \in \mathfrak{g}^*$ gives rise to a corresponding central quotient

$$B^\chi = A^G/(\mathfrak{g} - \chi(\mathfrak{g}))$$

where $\mathfrak{g} - \chi(\mathfrak{g})$ is a defined above. The B^χ may be considered as analogs of the minimal primitive quotients of enveloping algebras.

In this paper we give a fairly exhaustive description of the properties of B^χ. That is, we exhibit

(1) when B^χ is simple;
(2) when B^χ has finite global dimension;
(3) the various classical dimensions associated to B^χ : Krull-dimension, GK-dimension, injective dimension and homological dimension;
(4) the lattice of primitive ideals in B^χ and the corresponding primitive quotients;
(5) the category of finite dimensional representations of B^χ.

At this point, to get the flavor of our results, the reader is advised to consult §11. In that section we explicitly work out the example given by the rings of twisted differential operators on the first Hirzebruch surface.

To study B^χ we have been using systematically methods that have been introduced in the case of enveloping algebras. Our first task was to obtain an analog of Duflo's theorem. To this end we consider the abelian Lie algebra $t \subset A^G$ given by $k\pi_1 + \cdots + k\pi_n$ where $\pi_i = x_i\partial_i$. We view $\mathfrak{t}$ as an analog of the Cartan subalgebra of a semisimple Lie algebra. In particular with every $\alpha \in \mathfrak{t}^*$ we associate a simple A^G-module $L(\alpha)$. Then one of our results states that an analog of Duflo's theorem is true (see §7.3) :

Theorem B. *Every primitive ideal in A^G is the annihilator of some $L(\alpha)$.*

This result makes it possible to study the primitive ideals in A^G (or equivalently in B^χ) by purely combinatorial means. The following theorem is an extract of Theorem 7.3.1.

Theorem C. (1) *B^χ has only a finite number of primitive ideals.*
(2) *Every primitive ideal in A^G is generated by its intersection with the symmetric algebra of $\mathfrak{t}$.*

To study other properties of B^χ we introduce an analog of the translation principle. That is we exhibit certain B^χ-$B^{\chi'}$-bimodules, denoted by $B^{\chi,\chi'}$ which define a Morita context between B^χ and $B^{\chi'}$. In the event that $B^{\chi',\chi}B^{\chi,\chi'} = B^{\chi'}$ we write $\chi \to \chi'$. This defines a transitive relation on $\mathfrak{g}^*$. If $\chi \to \chi'$ and $\chi' \to \chi$ then B^χ and $B^{\chi'}$ are Morita equivalent. In the enveloping algebra case $\chi \to \chi'$ would mean that the central character χ' is more singular than the central character χ.

If we think of $\to$ as $\geq$ then we can define the properties of minimality and maximality for elements of $\mathfrak{g}^*$. This yields the following result

Theorem D. (1) *B^χ is simple if and only if χ is minimal.*
(2) *B^χ has finite global dimension if and only if χ is maximal.*

The reader will undoubtly recognize the corresponding statements for enveloping algebras. See for example [9].

As indicated above we also study the various dimensions attached to B^χ. For Krull-dimension and GK-dimension this is relatively easy and follows from standard results in ring theory. The case of injective dimension is slightly harder. We use a method introduced by Levasseur in [12]. This method is based upon a beautiful result of Joseph and Gabber [11, Thm 9.11] stating that if M is a finitely generated module over an enveloping algebra U over an *algebraic* Lie algebra then the following inequality holds

$$2\,\mathrm{GKdim}\, M \geq \mathrm{GKdim}(U/\operatorname{Ann} M)$$

Again we give an analog of this result for A^G. Unfortunately we have only been able to do this when M is simple.

Theorem E. *Assume that M is a simple A^G-module. Then*

$$2\,\mathrm{GKdim}\, M \geq \mathrm{GKdim}(A^G/\dim M)$$

When we started out writing this paper we noticed that that most of the basic results can actually be proved in greater generality. So given a finte dimensional abelian Lie algebra $\mathfrak{t}$ we study in §3 and §4 a class of associative algebras A equipped with a Lie algebra homomorphism $\phi : \mathfrak{t} \to A$, such that the following conditions hold

(A1) A is a semi-simple $\mathfrak{t}$ module for the adjoint action of $\mathfrak{t}$ on A.

(A2) The non-zero weight spaces in A are generated by one element over the symmetric algebra of $\mathfrak{t}$.

The reader can easily verify that if A is the right hand side of (1.1) then A, A^G and B^χ all satisfy these properties. However there are many more rings for which (A1)(A2) holds. Consider for example the analogs of $U(\mathfrak{sl}_2)$ studied by S.P. Smith in [23]. These are algebras $A = k[H, E, F]$ with relations

$$[H, E] = E, \qquad [H, F] = -F, \qquad [E, F] = f(H)$$

where f is a fixed polynomial in one variable.

Such an algebra contains a certain central element Ω that can be considered as an analog of the Casimir element. For $\mathfrak{t}$ we take $kH + k\Omega$. Then (A1) and (A2) hold and as a consequence we can recover the results in [3][8][23][33] on these algebras. We also prove a few new results such as Prop. 5.4.1 and Cor. 5.4.3.

We close this introduction by mentioning a few things that are not covered in this paper. First of all Theorems C(1), D and E can be stated without reference to our specific hypotheses on X, so they should be generalized accordingly. When G a torus this is probably a fairly simple consequence of the Luna Slice Theorem. If G is a general reductive group then everything is wide open.

Furthermore there are some applications specific for toric varieties. Most notably the Bernstein-Beilinson theorem ("localization"). The naive generalization of this result fails [26], but it is still possible to obtain fairly precise information on the category $\mathcal{D}$-modules on a smooth toric variety, starting from a ring of twisted global differential operators.

2. Notations and conventions

Most notations are introduced locally. The few global notations we use are given below.

If I is an ideal in a commutative ring R then $V(I)$ stands for the closed subscheme $\operatorname{Spec} R/I$ of $\operatorname{Spec} R$. Conversely if V is a closed subscheme of $\operatorname{Spec} R$ then $I(V)$ denotes the corresponding ideal.

If X is an object graded by a group G then for $g \in G$, X_g will be the part of degree g in X and $X(g)$ will be X, but with the grading shifted by g.

If G is a torus then $X(G)$, $Y(G)$ denote respectively the character group of G and the group of one-parameters of G. By $X(G)_\mathbb{Q}$, $Y(G)_\mathbb{Q}$ we denote the same groups but tensored by $\mathbb{Q}$.

We also mention our slightly unconventional way of defining the path algebra of a quiver. That is, we write a path $\xrightarrow{a}\xrightarrow{b}$ as ba. This has the effect that representations of quivers correspond to *left* modules over their path algebras.

Throughout this paper "iff" will mean "if and only if".

3. A certain class of rings

In this section we discuss a certain class of rings whose modules and homological properties may sometimes be described by combinatorial means. Examples will be given in subsequent sections. Here we discuss things in greater generality than is needed afterwards, in the hope that the results may be useful elsewhere.

3.1. **Generalities.** Let k be an algebraically closed base field of characteristic zero and let let A be a k-algebra. Let $\mathfrak{t}$ be a finite dimensional abelian Lie algebra and let $\phi : \mathfrak{t} \to A$ be a map of k-vector spaces whose image consists of commuting elements. Put $D = S\mathfrak{t}$, the symmetric algebra of $\mathfrak{t}$. We identify $\operatorname{Spec} D$ with $\mathfrak{t}^*$. We also extend ϕ to a k-algebra map $D \to A$, also denoted by ϕ. In the sequel when we consider the induced D-action on A-modules, we will usually suppress ϕ in the notation.

Let $\mathfrak{t}$ act on A by the adjoint action. That is $\pi \in \mathfrak{t}$ acts as $[\phi(\pi), -]$. Throughout we make the following assumptions.

(A1) A is a semi-simple $\mathfrak{t}$-module.
(A2) The non-zero weight spaces in A are generated on the left (and hence on the right) by one element over D.

Remark 3.1.1. In the above setting we could of course replace $\mathfrak{t}$ by its image in A. However in the sequel we will also be interested in quotients of A in which the image of $\mathfrak{t}$ will be different. Therefore we prefer to keep $\mathfrak{t}$ as a separate entity.

From (A1) we obtain a weight space decomposition

$$A = \bigoplus_{\alpha \in \mathfrak{t}^*} A_\alpha$$

which is easily seen to be a $\mathfrak{t}^*$-grading. Furthermore, every two sided ideal of A is graded for this grading. Note that (A2) implies that D maps onto A_0.

We will denote the category of $\mathfrak{t}^*$-graded (left) A-modules by A-Gr. Let M be in A-Gr. That is $M = \bigoplus_{\alpha \in \mathfrak{t}^*} M_\alpha$. We define a right action of D on M by

$$m\pi = (\pi - \alpha(\pi))m \tag{3.1}$$

for $\pi \in \mathfrak{t}$, $\alpha \in \mathfrak{t}^*$, $m \in M_\alpha$.

This definition makes M into a graded A-D-bimodule. The following results are easily proved.

Proposition 3.1.2. (1) *Eq.* (3.1) *defines an equivalence between* A-Gr *and the full subcategory of* A-D-mod *consisting of those modules which are semi-simple for the induced adjoint action of* $\mathfrak{t}$.
(2) *If* $M, N \in A$-Gr *then the induced* D-D*-bimodule structure on* $\operatorname{Hom}_{A\text{-Gr}}(M, N)$ *is central. In particular if* $M = N$ *then* D *is mapped to the center of* $\operatorname{End}_{A\text{-Gr}}(M)$.

If $\alpha \in \mathfrak{t}^*$ then we denote by m_α the corresponding maximal ideal in D.

Definition 3.1.3. Let $p \geq 1$. Then $\mathcal{O}^{(p)}$ is the full subcategory of A-mod consisting of those objects which are quotients of $\bigoplus_{\alpha \in \mathfrak{t}^*} (D/m_\alpha^p)$ as left D-modules. We also put $\mathcal{O}^{(\infty)} = \bigcup_{p \geq 1} \mathcal{O}^{(p)}$.

The following is clear.

Proposition 3.1.4. $\mathcal{O}^{(\infty)}$ *contains the category of finite-dimensional* A*-representations.*

Remark 3.1.5. One should think of $\mathcal{O}^{(1)}$ as a kind of $\mathcal{O}$-category by analogy with the usual definition in the case that A is the enveloping algebra of a semi-simple Lie algebra and $\mathfrak{t}$ is a Cartan subalgebra (note that this situation is *not* covered by our assumptions (A1)(A2)). However there are some differences. According to the definition in [5] objects in $\mathcal{O}$ are assumed to be finitely generated and to be locally finite for the action a fixed Borel subalgebra containing $\mathfrak{t}$. However the assumption of finite generation is not very essential, and in general there will be no good analog of a Borel subalgebra.

If $M \in \mathcal{O}^{(p)}$ then $M = \bigoplus_{\alpha\in\mathfrak{t}^*} M_\alpha$ where

$$M_\alpha = \{x \in M \mid m_\alpha^p x = 0\} \tag{3.2}$$

It is easy to verify that this defines a $\mathfrak{t}^*$-grading on M, compatible with the $\mathfrak{t}^*$-grading on A defined above. Furthermore if $M, N \in \mathcal{O}^{(p)}$ then

$$\operatorname{Hom}_{A\text{-mod}}(M,N) = \operatorname{Hom}_{A\text{-Gr}}(M,N)$$

Hence $\mathcal{O}^{(p)}$ may be considered as a full subcategory of A-Gr, but one should note that $\mathcal{O}^{(p)}$ is in general not stable under the shift functor $M \mapsto M(\alpha)$, $\alpha \in \mathfrak{t}^*$.

Since $\mathcal{O}^{(\infty)} = \bigcup_p \mathcal{O}^{(p)}$, all these considerations carry over to $\mathcal{O}^{(\infty)}$.

Remark 3.1.6. The fact that $\mathcal{O}^{(p)} \subset A$-Gr may lead to some confusion since now some objects will be equipped with two natural gradings. This is for example the case with $M = A/Am_\alpha$. On the one hand it is a quotient of A by a graded left ideal, so it inherits the grading on A. On the other hand M is in $\mathcal{O}^{(1)}$ so it is graded by (3.2). It is easy to see that these two gradings are different. For M they are still related by a shift but this is not true in general as one sees by considering the module $A/Am_\alpha \oplus A/Am_\beta$ with $\alpha \neq \beta$.

Fortunately it will usually be clear from the context which grading is being used. We accept as a rule that objects which are in $\mathcal{O}^{(p)}$ for some p are graded by (3.2), unless otherwise specified.

If $M \in A$-Gr then we define the support of M as

$$\operatorname{Supp} M = \{\alpha \in \mathfrak{t}^* \mid M_\alpha \neq 0\}$$

It is easy to see that if $M \in \mathcal{O}^{(p)}$ then $\operatorname{Supp} M \subset V(\ker\phi) \subset \operatorname{Spec} D = \mathfrak{t}^*$.

We define for $\alpha \in V(\ker\phi)$

$$M^{(p)}(\alpha) = A/Am_\alpha^p$$

Clearly $M^{(p)}(\alpha) \in \mathcal{O}^{(p)}$.

Proposition 3.1.7. *Let $\alpha_1, \alpha_2, \alpha_3, \alpha \in V(\ker\phi)$*

(1) *Let $M \in \mathcal{O}^{(p)}$. Then*

$$\operatorname{Hom}_A(M^{(p)}(\alpha), M) = M_\alpha$$

(2) *$M^{(p)}(\alpha)$ is projective in $\mathcal{O}^{(p)}$.*
(3) *$M^{(p)}(\alpha)$ has a unique simple quotient, denoted by $L(\alpha)$, which depends only on α, and not on p, and which lies in $\mathcal{O}^{(1)}$.*
(4) *All simple objects in $\mathcal{O}^{(p)}$ are of the form $L(\alpha)$.*
(5) *One has* $\dim M^{(1)}(\alpha)_{\alpha_1} \leq 1$ *and* $\dim L(\alpha)_{\alpha_1} \leq 1$.
(6) *The following are equivalent*
   (a) $L(\alpha_1) \cong L(\alpha_2)$

(b) $\operatorname{Supp} L(\alpha_1) \cap \operatorname{Supp} L(\alpha_2) \neq \emptyset$.
(c) $M^{(p)}(\alpha_1) \cong M^{(p)}(\alpha_2)$

(7) *One has identifications*

$$\begin{aligned}\operatorname{Hom}(M^{(p)}(\alpha_1), M^{(p)}(\alpha_2)) &= (A/Am^p_{\alpha_2})_{\alpha_1}\\ &= A_{\alpha_1-\alpha_2}/A_{\alpha_1-\alpha_2}m^p_{\alpha_2}\end{aligned}$$

(8) *The composition*

$$\operatorname{Hom}(M^{(p)}(\alpha_2), M^{(p)}(\alpha_3)) \times \operatorname{Hom}(M^{(p)}(\alpha_1), M^{(p)}(\alpha_2)) \to \operatorname{Hom}(M^{(p)}(\alpha_1), M^{(p)}(\alpha_3))$$

corresponds under the identification given by (7) to

$$A_{\alpha_2-\alpha_3}/A_{\alpha_2-\alpha_3}m^p_{\alpha_3} \times A_{\alpha_1-\alpha_2}/A_{\alpha_1-\alpha_2}m^p_{\alpha_2} \to A_{\alpha_1-\alpha_3}/A_{\alpha_1-\alpha_3}m^p_{\alpha_3} : (\bar{a}, \bar{b}) \mapsto \overline{ba}$$

(9) *The natural map*

$$D \to \operatorname{End}(M^{(p)}(\alpha))$$

given by Proposition 3.1.2(2), corresponds under the identification given by (7)

$$\operatorname{End}(M^{(p)}(\alpha)) = A_0/A_0 m^p_\alpha$$

to

$$\mathfrak{t} \to A_0/A_0m^p_\alpha : \pi \mapsto \overline{\phi(\pi - \alpha(\pi))}$$

Proof. (1) Sending f to $f(\bar{1})$ defines an identification between

$$\operatorname{Hom}_A(A/Am^p_\alpha, M)$$

and

$$\{x \in M \mid m^p_\alpha x = 0\}$$

which is precisely M_α.

(2) It follows from (1) that the functor $\operatorname{Hom}(M^{(p)}(\alpha), -)$ is exact. This implies that $M^{(p)}(\alpha)$ is projective.

(3) We have to show that $M^{(p)}(\alpha)$ has a unique maximal submodule. This amounts to showing that the sum of all proper submodules of $M^{(p)}(\alpha)$ is a proper submodule. Now $M^{(p)}(\alpha)_\alpha = A_0/A_0m^p_\alpha$. Hence $M^{(p)}(\alpha)$ is generated in degree α. This means that if $M \subset M^{(p)}(\alpha)$ is a submodule then M is a proper submodule if and only if M_α is a proper submodule. Now $A_0/A_0m^p_\alpha$ is a quotient of D/m^p_α which is local. Hence the sum of proper submodules of $M^{(p)}(\alpha)_\alpha$ is proper. This proves the existence of $L(\alpha)$.

We have furthermore a surjective map $M^{(p)}(\alpha) \to M^{(1)}(\alpha)$. Hence the unique simple quotient of $M^{(1)}(\alpha)$ is also the unique simple quotient of $M^{(p)}(\alpha)$. This shows that $L(\alpha)$ does not depend upon p.

(4) Let X be a simple object in $\mathcal{O}^{(p)}$. Then there is some α such that $X_\alpha \neq 0$. By (1) this means that there is a non-zero map $M^{(p)}(\alpha) \to X$. Since X is simple this map is surjective. But then by (3), $X = L(\alpha)$.

(5) The result for $M^{(1)}(\alpha)$ follows from assumption (A2) and for $L(\alpha)$ from the fact that $L(\alpha)$ is a quotient of $M^{(1)}(\alpha)$.

(6) It follows from (2) and (3) that $M^{(p)}(\alpha)$ is a projective cover of $L(\alpha)$ in $\mathcal{O}^{(p)}$. Hence (6a) iff (6c).

The proof of (6a) iff (6b) is similar to the proof of (4). Assume that $\alpha \in \operatorname{Supp} L(\alpha_1) \cap \operatorname{Supp} L(\alpha_2)$. Then according to (1), there are surjective maps

$M^{(p)}(\alpha) \to L(\alpha_1)$, $M^{(p)}(\alpha) \to L(\alpha_2)$. By (3) this implies that $L(\alpha_1) \cong L(\alpha_2)$.

(7) This follows from (1).

(8) If $b \in A_{\alpha_1-\alpha_2}$ then the corresponding map $A/Am^p_{\alpha_1} \to A/Am^p_{\alpha_2}$ is given by $\bar{a} \mapsto \overline{ab}$. This implies (8).

(9) Let $a \in A_0$. Then under the identification given by (7), a corresponds to the map

$$A/Am^p_\alpha \to A/Am^p_\alpha : \bar{b} \mapsto \overline{ba} \tag{3.3}$$

Now let $\pi \in \mathfrak{t}$. Then π corresponds to

$$A/Am^p_\alpha \to A/Am^p_\alpha : \bar{b} \mapsto \bar{b} \cdot \pi$$

where as usual the right D-module structure on A/Am^p_α is given by (3.1). That is if $\bar{b} \in (A/Am^p_\alpha)_{\alpha_1} = A_{\alpha_1-\alpha}/A_{\alpha_1-\alpha}m^p_\alpha$ then

$$\begin{aligned}\bar{b} \cdot \pi = (\pi - \alpha_1(\pi))\bar{b} = \overline{(\pi - \alpha_1(\pi))b} = \overline{b(\pi + (\alpha_1(\pi) - \alpha(\pi)) - \alpha_1(\pi))} \\ = \overline{b(\pi - \alpha(\pi))}\end{aligned}$$

Comparing this with (3.3) yields (9). $\square$

If $\alpha, \beta \in V(\ker \phi)$ then we put $\alpha \iff \beta$ iff $L(\alpha) \cong L(\beta)$ (or equivalently, iff $\beta \in \operatorname{Supp} L(\alpha)$).

Lemma 3.1.8. *If $M \in \mathcal{O}^{(p)}$ then $\operatorname{Supp} M$ is a union of equivalence classes for $\iff$.*

Proof. Assume that $M_\alpha \neq 0$. Then by Proposition 3.1.7(1) there is a non-zero map $M^{(p)}(\alpha) \to M$. Let N be the image of this map. Then by Proposition 3.1.7(3) N has $L(\alpha)$ as a quotient. Hence $L(\alpha)$ is a subquotient of M. This proves the lemma. $\square$

From Proposition 3.1.7(5) it follows that submodules of $M^{(1)}(\alpha)$ may be described by subsets of $\operatorname{Supp} M^{(1)}(\alpha)$. To describe such subsets we introduce a new relation. If $\alpha, \beta, \gamma \in V(\ker \phi)$ then we put $\beta \underset{\alpha}{\Rightarrow} \gamma$ iff $A_{\gamma-\beta} M^{(1)}(\alpha)_\beta \neq 0$. Then we have the following.

Lemma 3.1.9. *Let $\alpha, \beta, \gamma \in V(\ker \phi)$.*

(1) *$\underset{\alpha}{\Rightarrow}$ is a transitive relation on $V(\ker \phi)$.*

(2) *$\beta \underset{\alpha}{\Rightarrow} \gamma$ implies $\beta, \gamma \in \operatorname{Supp} M^{(1)}(\alpha)$ and $\gamma - \beta \in \operatorname{Supp} A$*

(3) *One has $\beta \underset{\alpha}{\Rightarrow} \gamma$ iff $A_{\gamma-\beta}A_{\beta-\alpha} \not\subset A_{\gamma-\alpha}m_\alpha$.*

(4) *Submodules of $M^{(1)}(\alpha)$ correspond to $\underset{\alpha}{\Rightarrow}$ closed subsets of $\operatorname{Supp} M^{(1)}(\alpha)$.*

(5) *One has $\beta \underset{\alpha}{\Rightarrow} \gamma$ and $\gamma \underset{\alpha}{\Rightarrow} \beta$ if and only if $\beta \iff \gamma$ and $\beta, \gamma \in \operatorname{Supp} M^{(1)}(\alpha)$.*

(6) *One has $\beta \iff \gamma$ iff $A_{\beta-\gamma}A_{\gamma-\beta} \not\subset m_\beta A_0$.*

Proof. (1)-(4) are immediate, so we concentrate on (5). Suppose first that $\beta \iff \gamma$. By lemma 3.1.8, any submodule containing $M^{(1)}(\alpha)_\beta$, must contain $M^{(1)}(\alpha)_\gamma$. Hence $\beta \underset{\alpha}{\Rightarrow} \gamma$, and $\gamma \underset{\alpha}{\Rightarrow} \beta$ holds by symmetry.

Assume now $\beta \underset{\alpha}{\Rightarrow} \gamma$ and $\gamma \underset{\alpha}{\Rightarrow} \beta$. Then by (2) β, $\gamma \in \operatorname{Supp} M^{(1)}(\alpha)$. Hence by Prop. 3.1.7(1) there are non-zero maps $M^{(1)}(\beta) \to M^{(1)}(\alpha)$ and $M^{(1)}(\gamma) \to M^{(1)}(\alpha)$ whose image is the same. Hence by Proposition 3.1.7(3), $L(\beta) = L(\gamma)$.

Now we prove (6). It follows from (2)(5) that $\beta \iff \gamma$ iff $\gamma \underset{\beta}{\Rightarrow} \beta$. The latter is equivalent with $A_{\beta-\gamma}A_{\gamma-\beta} \not\subset m_\beta A_0$ by (3). $\square$

The following technical result shows that the $\iff$ relation gives us some information about two sided ideals.

Proposition 3.1.10. *Assume that I is a two-sided ideal in A. Then $V(I_0)$, considered as a subset of $V(\ker\phi)$, is a union of equivalence classes for $\iff$.*

Proof. Let $\alpha \in V(I_0)$ and put $M = A/(Am_\alpha + I)$. Then $M_\alpha \neq 0$ and furthermore I_0 annihilates M. Thus $\alpha \in \operatorname{Supp} M \subset V(I_0)$. We then conclude by lemma 3.1.8 that $V(I_0)$ contains the equivalence class of α. $\square$

3.2. **Primitive ideals.** In this section we want to describe, under certain hypotheses, the primitive ideals of the rings we have introduced in §3.1. However we first prove the following technical result which holds in a somewhat greater generality.

Lemma 3.2.1. *Assume that $A = \oplus_{\alpha\in G} A_\alpha$ is a k-algebra graded by a group G. Assume furthermore that*

1. *A is graded prime and graded left noetherian.*
2. *A_0 is commutative and finitely generated over k.*
3. *For all $\alpha \in G$ there exists $u_\alpha \in A_\alpha$ such that*
$$A_\alpha = u_\alpha A_0 = A_0 u_\alpha$$

Then $\bigcap_{m\in\Omega(A_0)} Am = 0$, where $\Omega(A_0)$ denotes the set of maximal ideals of A_0.

Proof. In the proof we use some elementary notions from the theory of GK-dimension. We refer the reader to [11] for background.

Step 1. First of all we show that all $(A_\alpha)_{\alpha\in G}$ are homogeneous A_0-modules, and have the same dimension (we recall that a module M of GK-dimension t is homogeneous if it has no submodules of GK-dimension strictly smaller than t).

Assume that $\operatorname{GKdim} A_0 = t$. For all $\alpha \in G$ let I_α be the maximal left submodule of A_α of GKdim $< t$.

If $x \in A_0$ then $I_\alpha x \subset I_\alpha$, and hence I_α is a A_0-bimodule. Furthermore I_α is obviously the maximal *right* submodule of A_α having GKdim $< t$. Also $\operatorname{GKdim}(I_\alpha A_\beta) = \operatorname{GKdim}(I_\alpha u_\beta) < t$ and hence $I_\alpha A_\beta \subset I_{\alpha+\beta}$. Similarly $A_\beta I_\alpha \subset I_{\alpha+\beta}$. Therefore $I = \bigoplus_{\alpha\in G} I_\alpha$ is a graded two sided ideal in A.

Let J be the right annihilator of I. We claim that $J_0 \neq 0$. Since A is graded left noetherian one has $I = Ax_1 + \cdots + Ax_n$ where $x_i \in I_{\alpha_i}$. Then $J_0 = \bigcap_i \operatorname{Ann}_{A_0}(x_i)$. Now $\operatorname{GKdim}(A_0/\operatorname{Ann}_{A_0}(x_i)) < t$, and hence $\bigcap_i \operatorname{Ann}_{A_0}(x_i) \neq 0$, since by hypothesis, A_0 has dimension t.

So now $IJ = 0$ and, since A is graded prime, we obtain $I = 0$.

Step 2. Now we show that A_0 is semi-prime, and all $(A_\alpha)_{\alpha\in G}$ are isomorphic to semi-prime quotients of A_0 (as a left and as a right module). We need the following sublemma

Sublemma . Assume that R, S are commutative finitely generated k-algebras. Suppose furthermore that R, S are of the same dimension and homogeneous as modules over themselves. Let $\phi : R \to S$ be a surjective map. Then $\phi(\operatorname{rad} R) = \operatorname{rad} S$ where $\operatorname{rad}(-)$ denotes the nil radical.

Proof. If P is a minimal prime ideal of S then $\phi^{-1}(P)$ is a prime ideal in R such that $R/\phi^{-1}(P) \cong S/P$. Hence $\operatorname{GKdim} R/\phi^{-1}(P) = \operatorname{GKdim} S/P = \operatorname{GKdim} S = \operatorname{GKdim} R$ and therefore $\phi^{-1}(P)$ is a minimal prime of R. Hence $\operatorname{rad} R \subset \phi^{-1}(\operatorname{rad} S)$ which implies $\phi(\operatorname{rad} R) \subset \operatorname{rad} S$. So by dividing out $\operatorname{rad} R$, we may assume that R is semi-prime.

Now assume that $t \in R$ is regular, that is, not contained in any minimal prime. Suppose that $\phi(t)$ is contained in a minimal prime P of S. Then $t \in \phi^{-1}(P)$ which is a contradiction by what we have said in the previous paragraph. Hence $\phi(t)$ is regular. Let R', S' be resp. the localizations of R and S at all regular elements of R. Since R' is semi-prime and artinian, R' is a direct sum of fields. But then the same holds for for S'. Since $S \subset S'$ it follows that S' is semi-prime and we are done. □

For $\alpha \in G$ denote by $X_\alpha, Y_\alpha \subset A_0$ resp. the left and the right annihilator of u_α. Put $B_\alpha = A_0/X_\alpha$, $C_\alpha = A_0/Y_\alpha$. We consider A_α as $B_\alpha - C_\alpha$-bimodule which is left and right free of rank one. Putting $bu_\alpha = u_\alpha\theta(b)$ defines an isomorphism between B_α and C_α. Let $\delta : A_0 \to B_\alpha$, $\epsilon : A_0 \to C_\alpha$ be the quotient maps.

In the following computation the sublemma is used several times.

$$\begin{aligned}(\operatorname{rad} A_0)A_\alpha = \delta(\operatorname{rad} A_0)A_\alpha = (\operatorname{rad} B_\alpha)A_\alpha = A_\alpha\theta(\operatorname{rad} B_\alpha) = A_\alpha(\operatorname{rad} C_\alpha) = \\ A_\alpha\epsilon(\operatorname{rad} A_0) = A_\alpha(\operatorname{rad} A_0)\end{aligned}$$

Hence $(\operatorname{rad} A_0)A = A(\operatorname{rad} A_0)$ is a two sided ideal in A which is obviously nilpotent. Since A is graded prime this implies that $\operatorname{rad} A_0 = 0$. By the sublemma we deduce that B_α, C_α are semi-prime. Hence A_α is left and right isomorphic to a semi-prime quotient of A_0.

Step 3. The conclusion $\bigcap Am = 0$ now follows easily from step 2 and the nullstellensatz. □

Now we use again the notation of §3.1. So $A, D, \phi, \mathfrak{t}, \ldots$ will have their usual meaning, in particular A satisfies (A1)(A2).

If $\alpha \in V(\ker\phi)$ then we denote by $\langle\alpha\rangle$ the equivalence class for $\iff$ associated to α. We also put $J(\alpha) = \operatorname{Ann}_A L(\alpha)$. This is a primitive ideal in A. If $R \subset V(\ker\phi)$ then by $\bar{R}$ we denote the Zariski-closure of R.

Proposition 3.2.2. (1) *$J(\alpha)_0$ is semi-prime and $V(J(\alpha)_0) = \overline{\langle\alpha\rangle}$.*

(2) *Assume that for all $\beta \in \operatorname{Supp} A$ one has $I\left(\overline{(\langle\alpha\rangle + \beta) \cap \langle\alpha\rangle}\right) = I\left(\overline{\langle\alpha\rangle + \beta}\right) + I\left(\overline{\langle\alpha\rangle}\right)$. Then for all $\beta \in \operatorname{Supp} A$ one has $J(\alpha)_\beta = J(\alpha)_0 A_\beta + A_\beta J(\alpha)_0$. In particular $J(\alpha)$ is generated in degree zero.*

Proof. Let us write $J(\alpha) = \bigoplus_{\beta \in \operatorname{Supp} A} \phi(I_\beta)u_\beta$ where I_β is an ideal in D. Then we may take

$$I_\beta = \{x \in D \mid \forall\gamma \in \mathfrak{t}^* : xu_\beta L(\alpha)_\gamma = 0\}$$

Now $u_\beta L(\alpha)_\gamma$ is zero unless $\gamma \in \langle\alpha\rangle$, $\gamma + \beta \in \langle\alpha\rangle$ and in that case it is equal to $L(\alpha)_{\gamma+\beta}$. Hence

$$\begin{aligned} I_\beta &= \{x \in D \mid \forall\gamma \in \langle\alpha\rangle \cap (\langle\alpha\rangle - \beta) : x \in m_{\gamma+\beta}\} \\ &= \{x \in D \mid \forall\delta \in (\langle\alpha\rangle + \beta) \cap \langle\alpha\rangle : x \in m_\delta\} \\ &= I\left(\overline{(\langle\alpha\rangle + \beta) \cap \langle\alpha\rangle}\right) \end{aligned}$$

Now (1) follows by substituting $\beta = 0$.

Furthermore $J(\alpha)_\beta$ contains $I_0 u_\beta + u_\beta I_0$ and $u_\beta I_0 = I_0' u_\beta$ where $I_0' = I\left(\overline{\langle\alpha\rangle + \beta}\right)$.

By hypotheses $I_\beta = I\left(\overline{(\langle\alpha\rangle + \beta) \cap \langle\alpha\rangle}\right)$ is equal to $I_0 + I_0' = I\left(\overline{\langle\alpha\rangle}\right) + I\left(\overline{\langle\alpha\rangle + \beta}\right)$ which shows that $J(\alpha)_\beta = J(\alpha)_0 u_\beta + u_\beta J(\alpha)_0$. □

Corollary 3.2.3. *If $L(\alpha)$ is finite dimensional then $J(\alpha)$ is generated in degree zero.*

The following theorem, whose formulation is unfortunately somewhat technical, is the main result of this section.

Theorem 3.2.4. *Assume that*

(1) *A is graded left noetherian.*
(2) *The length of $M^{(1)}(\alpha)$ is bounded, independently of α.*
(3) *There are only a finite number of different $\overline{\langle\alpha\rangle}$ where α runs through $V(\ker\phi)$.*
(4) *For all $\alpha \in V(\ker\phi)$ and for all β in $\operatorname{Supp} A$ one has $I\left(\overline{\langle\alpha\rangle + \beta}\right) + I\left(\overline{\langle\alpha\rangle}\right) = I\left(\overline{(\langle\alpha\rangle + \beta) \cap \langle\alpha\rangle}\right)$.*

Then

(1) *every prime ideal in A is of the form $J(\alpha)$ for some $\alpha \in V(\ker\phi)$. Hence in particular it is primitive;*
(2) *there is a one-one correspondence between the regions $\overline{\langle\alpha\rangle}$, $\alpha \in V(\ker\phi)$ and the primitive ideals in A. The correspondence is given by associating $J(\alpha)$ to $\alpha \in V(\ker\phi)$.*

Proof. (2) is a direct consequence of (1) and Proposition 3.2.2. So we prove (1).

Without loss of generality we may assume that A is prime and that we have to show that $J(\alpha) = 0$ for some $\alpha \in V(\ker\phi)$. By lemma 3.2.1 we know that $0 = \bigcap_\alpha \operatorname{Ann}_A M^{(1)}(\alpha)$. By hypotheses the length of $M^{(1)}(\alpha)$ is uniformly bounded, say by N. Hence there is some product

$$J(\alpha_1) \cdots J(\alpha_n) \subset \operatorname{Ann}_A M^{(1)}(\alpha)$$

where $n \leq N$. So one has

$$0 = \bigcap_{i \in I} J(\alpha_{1,i}) \cdots J(\alpha_{n_i,i}) \tag{3.4}$$

where I is some index set and $n_i \leq N$. By Proposition 3.2.2 $J(\alpha)$ is determined by $\overline{\langle\alpha\rangle}$, so by hypotheses there are only a finite number of different $J(\alpha)$'s. This implies that I in (3.4) may be taken to be finite. But then (3.4) is only possible if some $J(\alpha_{i,j})$ is zero. □

Remark 3.2.5. Let $\mathfrak{g}$ be the solvable Lie algebra with basis t, y such that $[t, y] = y$. Let $A = U(\mathfrak{g})$, graded by y-degree. Then it is easily checked that hypotheses (2) and (3) of Theorem 3.2.4 and its conclusion all fail to hold. Likewise Theorems B and C from the introduction are false for A.

3.3. **Simplicity.** $A, D, \phi, \mathfrak{t}$ will be as before. A satisfies (A1)(A2).

In this section we prove the following criterion for simplicity of A.

Proposition 3.3.1. *Assume that A is a domain and that* $\operatorname{Supp} A$ *is a group. Then A is simple if and only if the equivalence classes for $\iff$ are Zariski dense in $V(\ker\phi)$.*

Proof. Assume first that A is simple. Then for all $\alpha \in V(\ker\phi)$ one has $J(\alpha) = 0$. Hence by Proposition 3.2.2 : $\overline{\langle\alpha\rangle} = V(\ker\phi)$.

Now we prove the converse. Thus we assume that all equivalence classes for $\iff$ are Zariski dense in $V(\ker\phi)$. Assume that $I \subset A$ is a non-trivial two-sided ideal. We recall that I is automatically graded. Thus some $I_\alpha \neq 0$ and so $I_0 \supset A_{-\alpha}I_\alpha \neq 0$. Assume $\alpha \in V(I_0)$. Then by Proposition 3.1.10, $V(\ker\phi) = \overline{\langle\alpha\rangle} \subset V(I_0)$, which is impossible. □

Remark 3.3.2. The hypotheses of Proposition 3.3.1 are somewhat unsatisfactory since they are not preserved under taking quotients. At the cost of using hypotheses which may be more difficult to verify one may obtain a simplicity result from Theorem 3.2.4. Indeed if the hypotheses (1)(2)(4) hold in that theorem, and (3) is replaced by

(3') All $\langle\alpha\rangle$ are Zariski dense in $V(\ker\phi)$.

then A is simple because A has no primitive ideals.

3.4. **Integrality.** $A, D, \phi, \mathfrak{t}$ will be as before. A satisfies (A1)(A2).

Proposition 3.4.1. *Assume that*

1. *A_0 is a domain;*
2. *$\forall\alpha \in \operatorname{Supp} A : A_\alpha$ is (left or right) free over A_0;*
3. *for all $\beta, \gamma \in \operatorname{Supp} A$ there exists $\alpha \in \mathfrak{t}^*$ such that $\alpha + \gamma \in \langle\alpha\rangle$, $\alpha + \beta + \gamma \in \langle\alpha\rangle$.*

Then A is a domain.

Proof. From (A2) it is easy to see that A_α is left *and* right free, generated by an element u_α. Putting $au_\alpha = u_\alpha\theta_\alpha(a)$ defines an automorphism θ_α of A_0. It is now easy to see that A is a domain if for all $\beta, \gamma \in \operatorname{Supp} A$ one has $u_\beta u_\gamma \neq 0$. Hence suppose $u_\beta u_\gamma = 0$. By (3) : $u_\beta u_\gamma L(\alpha)_\alpha = u_\beta L(\alpha)_{\alpha+\gamma} = L(\alpha)_{\alpha+\beta+\gamma}$ which yields a contradiction. □

3.5. **Homological properties.** We assume that $A, \phi, D, \mathfrak{t}, \cdots$ have their usual meaning and in particular A satisfies (A1)(A2). The following finiteness property, which is easily proved, will be implicitly used many times.

Proposition 3.5.1. *Let M, N be objects in $\mathcal{O}^{(p)}$, finitely generated as A-modules. Then*

1. *All $(M_\alpha)_{\alpha\in\mathfrak{t}^*}$ are finite dimensional over k.*
2. *$\operatorname{Hom}_A(M, N)$ is a finite dimensional k-vector space.*

In this section we let $S \subset \mathfrak{t}$ stand for an abelian group containing $\operatorname{Supp} A$. The full subcategory of A-Gr of those modules whose support lies in S will be denoted by A-gr.

In this section we fix $\Lambda \in \mathfrak{t}^*/S$ and we denote by $\mathcal{O}^{(p)}_\Lambda$ the full subcategory of $\mathcal{O}^{(p)}_\Lambda$ of objects whose support lies in Λ. Of course $\mathcal{O}^{(\infty)}_\Lambda = \bigcup \mathcal{O}^{(p)}_\Lambda$.

Clearly if $M_1 \in \mathcal{O}^{(p)}_{\Lambda_1}$, $M_2 \in \mathcal{O}^{(p)}_{\Lambda_2}$, $\Lambda_1 \neq \Lambda_2$ then $\operatorname{Hom}_A(M_1, M_2) = 0$. So in some sense $\mathcal{O}^{(p)} = \bigoplus_\Lambda \mathcal{O}^{(p)}$.

One easily sees that $M^{(p)}(\alpha)$, $L(\alpha)$ lie in $\mathcal{O}_\Lambda^{(p)}$ iff $\alpha \in \Lambda$. Furthermore $\mathcal{O}_\Lambda^{(p)}$ is non-trivial iff $\Lambda \cap V(\ker\phi) \neq \emptyset$.

If $\mathcal{O}_\Lambda^{(p)}$ contains only a finite number of simples, then its objects may be described combinatorially.

Proposition 3.5.2. *Assume that $\mathcal{O}_\Lambda^{(p)}$ contains only a finite number of non-isomorphic simple objects $L(\alpha_1), \dots, L(\alpha_d)$ (or equivalently "$\Longleftrightarrow$" has only a finite number of equivalence classes in $\Lambda \cap V(\ker\phi)$).*

Put $M_\Lambda^{(p)} = M^{(p)}(\alpha_1) \oplus \cdots \oplus M^{(p)}(\alpha_d)$. Then the functor

$$F^{(p)} : M \mapsto \operatorname{Hom}_A(M_\Lambda^{(p)}, M)$$

defines an equivalence between $\mathcal{O}_\Lambda^{(p)}$ and the category of left*-modules over the finite-dimensional algebra*

$$H_\Lambda^{(p)} = \operatorname{End}(M_\Lambda^{(p)})^{\mathrm{opp}}$$

Under this equivalence, finitely generated modules in A-mod *correspond to finite dimensional representations of $H_\Lambda^{(p)}$.*

The functors $F^{(p)}$ are compatible in the sense that $F^{(p)} \mid \mathcal{O}_\Lambda^{(p-1)} = F^{(p-1)}$.

Proof. $M^{(p)}$ is a faithfully projective generator. The result now follows from [2, Ch. II, Thm. 1.3] □

In the rest of this section we assume that the hypotheses of Proposition 3.5.2 are fulfilled. That is, we assume

(A3) $\mathcal{O}_\Lambda^{(1)}$ contains a finite number of simples given by $L(\alpha_1), \dots, L(\alpha_d)$.

Using Proposition 3.1.7(8), we may give an explicit form for $H_\Lambda^{(p)}$

$$H_\Lambda^{(p)} = \begin{pmatrix} \operatorname{End}(M^{(p)}(\alpha_1)) & \operatorname{Hom}(M^{(p)}(\alpha_2), M^{(p)}(\alpha_1)) & \cdots \\ \operatorname{Hom}(M^{(p)}(\alpha_1), M^{(p)}(\alpha_2)) & \operatorname{End}(M^{(p)}(\alpha_2)) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}^{\mathrm{opp}}$$

$$= \begin{pmatrix} \operatorname{End}(M^{(p)}(\alpha_1))^{\mathrm{opp}} & \operatorname{Hom}(M^{(p)}(\alpha_1), M^{(p)}(\alpha_2)) & \cdots \\ \operatorname{Hom}(M^{(p)}(\alpha_2), M^{(p)}(\alpha_1)) & \operatorname{End}(M^{(p)}(\alpha_2))^{\mathrm{opp}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} A_0/A_0 m_{\alpha_1}^p & A_{\alpha_1-\alpha_2}/A_{\alpha_1-\alpha_2} m_{\alpha_2}^p & \cdots \\ A_{\alpha_2-\alpha_1}/A_{\alpha_2-\alpha_1} m_{\alpha_1}^p & A_0/A_0 m_{\alpha_2}^p & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{3.5}$$

where the multiplication in (3.5) is the natural one.

Similarly we find for $M \in \mathcal{O}_\Lambda^{(p)}$

$$F^{(p)}(M) = \begin{pmatrix} M_{\alpha_1} \\ \vdots \\ M_{\alpha_d} \end{pmatrix} \tag{3.6}$$

where the action of (3.5) on (3.6) is again the natural one.

We also deduce from 3.1.7(9) that the natural map

$$\psi : D \to H_\Lambda^{(p)}$$

(from Proposition 3.1.2(2)) is given by

$$\pi \mapsto \begin{pmatrix} \overline{\phi(\pi - \alpha_1(\pi))} & & & \\ & \overline{\phi(\pi - \alpha_2(\pi))} & & \\ & & \ddots & \\ & & & \overline{\phi(\pi - \alpha_d(\pi))} \end{pmatrix} \tag{3.7}$$

for $\pi \in \mathfrak{t}^*$.

Let

$$H_\Lambda = \begin{pmatrix} A_0 & A_{\alpha_1 - \alpha_2} & \cdots & \\ A_{\alpha_2 - \alpha_1} & A_0 & \cdots & \\ \vdots & \vdots & \ddots & \\ & & & A_0 \end{pmatrix} \tag{3.8}$$

with a map $\psi : D \to H_\Lambda$ given by

$$\pi \mapsto \begin{pmatrix} \phi(\pi - \alpha_1(\pi)) & & \\ & \ddots & \\ & & \phi(\pi - \alpha_d(\pi)) \end{pmatrix} \tag{3.9}$$

for $\pi \in \mathfrak{t}^*$. Then it is easy to see that the image of D in H_Λ is central. Furthermore one easily computes that

$$H_\Lambda^{(p)} = D/m_0^p \otimes_D H_\Lambda$$

Remark 3.5.3. There is a slight abuse of notation here since H_Λ does not only depend on Λ, but also on the particular choice of $\alpha_1, \alpha_2, \dots, \alpha_d$. This is not the case for $H_\Lambda^{(p)}$.

Sending $M \in \mathcal{O}^{(p)}$ to $M \otimes_D D/m_0^{p-1}$ defines a functor $\mathcal{O}_\Lambda^{(p)} \to \mathcal{O}_\Lambda^{(p-1)}$ which sends $M_\Lambda^{(p)}$ to $M_\Lambda^{(p-1)}$. Therefore we obtain an algebra homomorphism $H_\Lambda^{(p)} \to H_\Lambda^{(p-1)}$ which is easily seen to coincide with the natural map

$$D/m_0^p \otimes_D H_\Lambda \to D/m_0^{p-1} \otimes H_\Lambda$$

We define

$$H_\Lambda^{(\infty)} = \varprojlim_p H_\Lambda^{(p)}$$

Hence $H_\Lambda^{(\infty)}$ is the completion of H_Λ at the ideal m_0.

Corollary 3.5.4. *H_Λ ($H_\Lambda^{(\infty)}$) is finitely generated as a module over D ($\hat{D}_{m_0}$). In particular H_Λ ($H_\Lambda^{(\infty)}$) is left and right Noetherian.*

Proof. This follows from the explicit descriptions of H_Λ and $H_\Lambda^{(\infty)}$ given above. □

Lemma 3.5.5. *$H_\Lambda^{(\infty)}$ is independent of the choice of $\alpha_1, \dots, \alpha_d$.*

Proof. Choose $\beta_1, \dots, \beta_d$ in such a way that for $i = 1, \dots, d : \alpha_i \iff \beta_i$ and put $P^{(p)} = M^{(p)}(\beta_1) \oplus \cdots M^{(p)}(\beta_d)$. Choose isomorphisms $\phi_i^{(1)} : M^{(1)}(\alpha_i) \to M^{(1)}(\beta_i)$. Since $M^{(p)}(\alpha_i)$ is projective in $\mathcal{O}^{(p)}$, the canonical maps $\mathrm{Hom}(M^{(p)}(\alpha_i), M^{(p)}(\beta_i)) \to \mathrm{Hom}(M^{(p-1)}(\alpha_i), M^{(p-1)}(\beta_i))$ are surjective. Therefore we may lift $\phi_i^{(1)}$ to compatible maps $\phi_i^{(p)} : M^{(p)}(\alpha_i) \to M^{(p)}(\beta_i)$. It follows that $\phi_i^{(p)}$ is compatible with the surjections $M^{(p)}(\alpha_i) \to L(\alpha_i)$, $M^{(p)}(\beta_i) \to L(\beta_i) \cong L(\alpha_i)$. Hence $\phi_i^{(p)}$ is an isomorphism for all i and for all p. Hence there are compatible isomorphisms $\mathrm{End}(P^{(p)}) \to \mathrm{End}(M_\Lambda^{(p)})$ which yield an isomorphism between the corresponding inverse limits. $\square$

Proposition 3.5.6. *Let $F^{(\infty)} : \mathcal{O}_\Lambda^{(\infty)} \to H_\Lambda^{(\infty)}$-mod be defined by $F^{(\infty)} \mid \mathcal{O}_\Lambda^{(p)} = F^{(p)}$. Then $F^{(\infty)}$ defines an equivalence between the full subcategory of finitely generated objects of $\mathcal{O}_\Lambda^{(\infty)}$ and the category of finite dimensional $H_\Lambda^{(\infty)}$-modules.*

Proof. This follows from Proposition 3.5.2. $\square$

In the rest of this section α will be a fixed element of Λ, unless otherwise specified. Let $M \in A$-gr. Using the right action of D on M as defined by (3.1) it makes sense to write $M/Mm_\alpha^p = M \otimes_D D/m_\alpha^p$. Using the fact that M has by definition its weights in S we see that $M/Mm_\alpha^p \in \mathcal{O}_\Lambda^{(p)}$. Hence we may define the following functor

$$F_\alpha^{(\infty)} : A\text{-gr} \to H_\Lambda^{(\infty)}\text{-mod} : M \mapsto \varprojlim_p F^{(p)}(M/Mm_\alpha^p)$$

Since

$$F^{(p)}(M/Mm_\alpha^p) = \begin{pmatrix} (M/Mm_\alpha^p)_{\alpha_1} \\ \vdots \\ (M/Mm_\alpha^p)_{\alpha_d} \end{pmatrix} = \begin{pmatrix} M_{\alpha_1-\alpha}/M_{\alpha_1-\alpha}m_\alpha^p \\ \vdots \\ M_{\alpha_d-\alpha}/M_{\alpha_d-\alpha}m_\alpha^p \end{pmatrix} = \begin{pmatrix} M_{\alpha_1-\alpha}/m_{\alpha_1}^p M_{\alpha_1-\alpha} \\ \vdots \\ M_{\alpha_d-\alpha}/m_{\alpha_d}^p M_{\alpha_d-\alpha} \end{pmatrix}$$

we obtain a more convenient description of $F_\alpha^{(\infty)}(M)$ as the completion of the left H_Λ-module.

$$F_\alpha(M) = \begin{pmatrix} M_{\alpha_1-\alpha} \\ \vdots \\ M_{\alpha_d-\alpha} \end{pmatrix} \tag{3.10}$$

at the ideal m_0 of D (which maps to H_Λ as given by (3.9)). This description yields the following proposition :

Proposition 3.5.7. *The functor $F_\alpha^{(\infty)}$ sends finitely generated modules in A-gr to finitely generated $H_\Lambda^{(\infty)}$-modules and furthermore $F_\alpha^{(\infty)}$ is exact on such modules.*

We also obtain

$$D/m_0^p \otimes_D F_\alpha^{(\infty)}(M) = F^{(p)}(M/Mm_\alpha^p)$$

By generalizing this to maps we obtain that the following diagram of functors is commutative

$$\begin{tikzcd} A\text{-gr} \arrow[r, "F_\alpha^{(\infty)}"] \arrow[d, "-\otimes_D D/m_\alpha^p"] & H_\Lambda^{(\infty)} \arrow[d, "D/m_0^p\otimes -"] \\ \mathcal{O}_\Lambda^{(p)} \arrow[r, "F^{(p)}"] & H_\Lambda^{(p)}\text{-mod} \end{tikzcd} \tag{3.11}$$

$F_\alpha^{(\infty)}$ is a functor, so for $M, N \in A$-gr there is a natural map

$$\mathrm{Hom}_{A\text{-gr}}(M,N) \to \mathrm{Hom}_{H_\Lambda^{(\infty)}}(F_\alpha^{(\infty)}(M), F_\alpha^{(\infty)}(N)) \tag{3.12}$$

By Proposition 3.1.2 the left hand side of (3.12) is a central D-bimodule.

For all $\alpha \in \mathfrak{t}^*$ let $\hat{D}_\alpha$ denote the completion of D at the maximal ideal m_α. Then $\hat{D}_0$ maps to the center of $H_\Lambda^{(p)}$ and hence the right hand side is in a natural way a central $\hat{D}_0$-bimodule.

Proposition 3.5.8. *Assume that A is graded left noetherian and that M, N are finitely generated objects in A-gr. Then*

(1) *The map in* (3.12) *is continuous if we equip the left hand side with the m_α-adic topology, and the right hand side with the m_0-adic topology.*
(2) *Completing the map* (3.12) *with respect to the above topologies yields an isomorphism between* $\operatorname{Hom}_{A\text{-gr}}(M,N)\otimes_D \hat{D}_\alpha$ *and* $\operatorname{Hom}_{H_\Lambda^{(\infty)}}(F_\alpha^{(\infty)}(M), F_\alpha^{(\infty)}(N))$

Proof. For $l \geq 0$ let

$$K_l = \{f \in \operatorname{Hom}_{A\text{-gr}}(M,N) \mid f(M) \subset Nm_\alpha^l\}$$

Then $(K_l)_l$ defines a filtration on $\operatorname{Hom}_{A\text{-gr}}(M,N)$ which is cofinal with the m_α-adic filtration. To see this write M as a quotient of a finitely generated graded free A-module F. Then $\operatorname{Hom}_{A\text{-gr}}(M,N)$ embeds in $\operatorname{Hom}_{A\text{-gr}}(F,N)$ and

$$\begin{aligned} K_l &= \operatorname{Hom}_{A\text{-gr}}(M,N) \cap \operatorname{Hom}_{A\text{-gr}}(F, Nm_\alpha^l) \\ &= \operatorname{Hom}_{A\text{-gr}}(M,N) \cap \operatorname{Hom}_{A\text{-gr}}(F,N)m_\alpha^l \end{aligned}$$

It now suffices to invoke the Artin-Rees lemma for D.

Similarly if we put

$$L_l = \{f \in \operatorname{Hom}_{H_\Lambda^{(\infty)}}(F_\alpha^{(\infty)}(M), F_\alpha^{(\infty)}(N)) \mid f(F_\alpha^{(\infty)}(M)) \subset m_0^l F^{(\infty)}(N)\}$$

then this filtration on $\operatorname{Hom}_{H_\Lambda^{(\infty)}}(F_\alpha^{(\infty)}(M), F_\alpha^{(\infty)}(N))$ is cofinal with the m_0-adic filtration.

The commutative diagram of functors (3.11) yields a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{A\text{-gr}}(M,N) & \xrightarrow{F_\alpha^{(\infty)}} & \operatorname{Hom}_{H_\Lambda^{(\infty)}}(F_\alpha^{(\infty)}(M), F_\alpha^{(\infty)}(N)) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_A(M/Mm_\alpha^p, N/Nm_\alpha^p) & \xrightarrow{F^{(p)}} & \operatorname{Hom}_{H_\Lambda^{(p)}}(F^{(p)}(M/Mm_\alpha^p), F^{(p)}(N/Nm_\alpha^p)) \end{array}$$

One easily sees that K_p is the kernel of the leftmost vertical map, whereas L_p is the kernel of the rightmost vertical map. Hence $F_\alpha^{(\infty)}(K_l) \subset L_l$ and therefore $F_\alpha^{(\infty)}$ is continuous.

Since M/Mm_α^p, $N/Nm_\alpha^p \in \mathcal{O}_\Lambda^{(p)}$, we know that $F^{(p)}$ is an isomorphism for all p. Hence it suffices to show that the induced maps

$$\operatorname{Hom}_{A\text{-gr}}(M,N)\hat{} \to \varprojlim_p \operatorname{Hom}(M/Mm_\alpha^p, N/Nm_\alpha^p)$$

and

$$\operatorname{Hom}_{H_\Lambda^{(\infty)}}(F_\alpha^{(\infty)}(M), F_\alpha^{(\infty)}(N)) \to \varprojlim_p \operatorname{Hom}_{H_\Lambda^{(p)}}(F^{(p)}(M/Mm_\alpha^p), F^{(p)}(N/Nm_\alpha^p))$$

are isomorphisms. The first isomorphism follows easily by replacing M with a presentation $F_1 \to F_0$ where the F's are finitely generated graded free A-modules. It then suffices to look at the case $M = A(s)$; $s \in S$, which is trivial. The second isomorphism is proved in a similar way. $\square$

Proposition 3.5.9. *Assume that A is graded left Noetherian. Then $F_\alpha^{(\infty)}$ sends finitely generated graded projectives to projectives.*

Proof. Since $F_\alpha^{(\infty)}$ is compatible with direct sums, it suffices to show that $F_\alpha^{(\infty)}(A(s))$, $s \in S$ is projective.

Now $F_\alpha^{(\infty)}(A(s)) = \varprojlim_p F^{(p)}(M^{(p)}(\alpha - s))$, and, as in the proof of lemma 3.5.5 there exists some α_i and compatible isomorphisms $M^{(p)}(\alpha_i) \to M^{(p)}(\alpha - s)$ which yield an isomorphism between $\varprojlim_p F_\alpha^{(p)}(M^{(p)}(\alpha_i))$ and $F_\alpha(A(s))$.

Let $e_i^{(p)} \in H_\Lambda^{(p)}$ be the projection $M_\Lambda^{(p)}$ on $M^{(p)}(\alpha_i)$ and put $e_i = \varprojlim e_i^{(p)}$. Then $F_\alpha^{(p)}(M^{(p)}(\alpha_i)) = H_\Lambda^{(p)} e_i^{(p)}$ and correspondingly $\varprojlim_p F_\alpha^{(p)}(M^{(p)}(\alpha_i)) = H_\Lambda^{(\infty)} e_i$ which is projective. $\square$

Corollary 3.5.10. *Assume that A is graded left Noetherian. Let M, N be finitely generated objects in A-gr. Then there is a natural isomorphism*

$$\operatorname{Ext}^i_{A\text{-gr}}(M, N) \otimes_D \hat{D}_\alpha \cong \operatorname{Ext}^i_{H_\Lambda^{(\infty)}}(F_\alpha^{(\infty)}(M), F_\alpha^{(\infty)}(N))) \tag{3.13}$$

Proof. One replaces M by a resolution $P^\cdot$, consisting of finitely generated graded projective modules. Then by Proposition 3.5.7 and 3.5.9 $F_\alpha^{(\infty)}(P^\cdot)$ is a projective resolution of $F_\alpha(M)$, and (3.13) easily follows. $\square$

Corollary 3.5.11. *Assume that A is graded left Noetherian.*

(1) *One has* $\operatorname{gl\,dim} H_\Lambda^{(\infty)} \le \text{gr. } \operatorname{gl\,dim} A$.
(2) *Assume that for all $\Gamma \in \mathfrak{t}^*/S$ there exist only a finite number of non-isomorphic simples in $\mathcal{O}_\Gamma^{(1)}$. Then*

$$\text{gr. } \operatorname{gl\,dim} A = \max_\Gamma \operatorname{gl\,dim} H_\Gamma^{(\infty)}$$

Proof. (1) Let $q = \text{gr. } \operatorname{gl\,dim} A$. Since $H_\Lambda^{(\infty)}$ is finite as a module over a commutative ring, it suffices to show that if X, Y are simple $H_\Lambda^{(\infty)}$-modules then

$$\operatorname{Ext}^m_{H_\Lambda^{(\infty)}}(X, Y) = 0 \tag{3.14}$$

for $m > q$.

Now there exist i, j such that

$$F^{(1)}(L(\alpha_i)) = X$$
$$F^{(1)}(L(\alpha_j)) = Y$$

Let as before $\alpha \in \Lambda$. Obviously $L(\alpha_i)(\alpha) \in A$-gr ($L(\alpha_i)$ is shifted by $\alpha \in \mathfrak{t}^*$). Furthermore $L(\alpha_i)(\alpha) m_\alpha = 0$ (we recall once again that the right D-action is determined by the grading).

Therefore

$$F_\alpha^{(\infty)}(L(\alpha_i)(\alpha)) = F^{(1)}(L(\alpha_i)) = X$$
$$F_\alpha^{(\infty)}(L(\alpha_j)(\alpha)) = F^{(1)}(L(\alpha_j)) = Y$$

The result now follows from cor. 3.5.10

(2) Let $n = \max_\Gamma \operatorname{gl}\dim H_\Gamma^{(\infty)}$. It follows from (1) that gr. $\operatorname{gl}\dim A \geq n$. So we only have to prove the opposite inequality. By cor. 3.5.10 we have for all finitely generated $M, N \in A$-gr and all $\alpha \in V(\ker \phi)$ that

$$\operatorname{Ext}^m_{A\text{-gr}}(M,N) \otimes_D \hat{D}_\alpha = 0$$

for $m > n$. Hence $\operatorname{Ext}^m_{A\text{-gr}}(M,N) = 0$ for $m > n$ and the conclusion follows. □

Remark 3.5.12. It follows from [20, A.II.8.2] that if A is graded by a torsion free abelian group of rank n then

$$\text{gr. } \operatorname{gl}\dim A \leq \operatorname{gl}\dim A \leq \text{gr. } \operatorname{gl}\dim A + n$$

So in this way corollary 3.5.11 yields a criterion for A to have finite global dimension, but the exact value of this global dimension remains unclear. Nevertheless we conjecture that under reasonable extra hypotheses our rings will have the property that $\operatorname{gl}\dim A =$ gr. $\operatorname{gl}\dim A$.

4. Some constructions

In this section we keep the notations of §3. We study the behavior of the ring A under some standard ring theoretical constructions. These will be used when we apply the results obtained so far to rings of differential operators. Since some of the constructions below may appear unmotivated, the reader is advised to skim this section, and to come back to it later, when needed.

To stress the dependency of our objects on the ring A we will sometimes use the notations $\mathfrak{t}_A$, ϕ_A, $\mathcal{O}^{(p)}_{\Lambda,A}$, $H^{(p)}_{\Lambda,A}$, etc.... We assume that such notations are self explanatory.

We recall that S(or S_A) $\subset \mathfrak{t}$ is an abelian group containing $\operatorname{Supp} A$.

4.1. **Tensor products.** Let $(\mathfrak{t}_A, \phi_A, A)$, $(\mathfrak{t}_B, \phi_B, B)$ be as in §3 (in particular they satisfy (A1)(A2)). We put $C = A \otimes_k B$, $\mathfrak{t}_C = \mathfrak{t}_A \oplus \mathfrak{t}_B$ and we define $\phi_C : \mathfrak{t}_C \to C$ by $\phi_C \mid \mathfrak{t}_A = \phi_A \otimes 1$, $\phi_C \mid \mathfrak{t}_B = 1 \otimes \phi_B$. It is clear that $(\mathfrak{t}_C, \phi_C, C)$, again satisfies (A1)(A2).

Below we will write $\alpha \in \mathfrak{t}_C^*$ as a couple (α_1, α_2), where $\alpha_1 \in \mathfrak{t}_A^*$, $\alpha_2 \in \mathfrak{t}_B^*$.

Proposition 4.1.1. *Let $\alpha, \beta, \gamma \in \mathfrak{t}_C^*$. Then*

(1) *$M^{(1)}(\alpha) = M^{(1)}(\alpha_1) \otimes_k M^{(1)}(\alpha_2)$;*
(2) *$L(\alpha) = L(\alpha_1) \otimes_k L(\alpha_2)$;*
(3) *one has $\beta \underset{\alpha}{\Rightarrow} \gamma$ iff $\beta_1 \underset{\alpha_1}{\Rightarrow} \gamma_1$ and $\beta_2 \underset{\alpha_2}{\Rightarrow} \gamma_2$;*
(4) *similarly $\beta \iff \gamma$ iff $\beta_1 \iff \gamma_1$ and $\beta_2 \iff \gamma_2$;*
(5) *Put $S_C = S_A \oplus S_B$. Choose $\Lambda \in \mathfrak{t}_C^*/S_C$. Then $\Lambda = \Lambda_1 \oplus \Lambda_2$ where $\Lambda_1 \in \mathfrak{t}_A^*/S_A$, $\Lambda_2 \in \mathfrak{t}_B^*/S_B$. Assume that Λ_1, Λ_2 satisfy (A3) (of section 3.5). Then $H_\Lambda = H_{\Lambda_1} \otimes H_{\Lambda_2}$ and consequently $H_\Lambda^{(\infty)} = H_{\Lambda_1}^{(\infty)} \hat{\otimes}_k H_{\Lambda_2}^{(\infty)}$ where $\hat{\otimes}$ denotes the completed tensor product.*

Proof. (1) This is clear from the definition.

(2) There is a map $M^{(1)}(\alpha) \cong M^{(1)}(\alpha_1) \otimes_k M^{(1)}(\alpha_2) \to L(\alpha_1) \otimes_k L(\alpha_2)$. According to Proposition 3.1.7(3) this implies the existence of a non-zero map $L(\alpha_1) \otimes_k L(\alpha_2) \to L(\alpha)$, hence it suffices to show that $L(\alpha_1) \otimes_k L(\alpha_2)$ is simple. Since $L(\alpha_1) \otimes_k L(\alpha_2)$ is in $\mathcal{O}^{(1)}$, any non-trivial submodule

$M \subset L(\alpha_1) \otimes_k L(\alpha_2)$ is automatically graded and hence some M_β is non-zero. Therefore it suffices to show that $L(\alpha_1) \otimes L(\alpha_2)$ is generated by $L(\alpha_1)_{\beta_1} \otimes L(\alpha_2)_{\beta_2}$, which is clear.

(3) We use the criterion from Lemma 3.1.9. That is, $\beta \underset{\alpha}{\Rightarrow} \gamma$ iff $C_{\gamma-\beta}C_{\beta-\alpha}$ has non-zero image in $C_{\gamma-\alpha}/C_{\gamma-\alpha}m_\alpha$. This is equivalent with the image of $A_{\gamma_1-\beta_1}A_{\beta_1-\alpha_1} \otimes B_{\gamma_2-\beta_2}B_{\beta_2-\alpha_2}$ being non-zero in $A_{\gamma_1-\alpha_1}/A_{\gamma_1-\alpha_1}m_{\alpha_1} \otimes B_{\gamma_2-\alpha_2}/B_{\gamma_2-\alpha_2}m_{\alpha_2}$, which in turn is equivalent with $\beta_1 \underset{\alpha_1}{\Rightarrow} \gamma_1$ and $\beta_2 \underset{\alpha_2}{\Rightarrow} \gamma_2$.

(4) This follows from (2), or from (3).

(5) If the simple objects in $\mathcal{O}^{(1)}_{\Lambda_1}$ and $\mathcal{O}^{(1)}_{\Lambda_2}$ are respectively $L(\alpha_1), \dots, L(\alpha_d)$ and $L(\beta_1), \dots, L(\beta_e)$ then by (4) the simples in $\mathcal{O}^{(1)}_{\Lambda}$ are $L(\alpha_1, \beta_1), \dots, L(\alpha_d, \beta_e)$. The formula for H_Λ now follows from the fact that $C_{(\alpha_i,\beta_j)} = A_{\alpha_i} \otimes B_{\beta_j}$, and the formula for $H^{(\infty)}_\Lambda$ follows by completing. □

4.2. **Quotients.** We assume that $(\mathfrak{t}, \phi, A)$ satisfies (A1)(A2). Let $c \in \mathfrak{t}$ be such that $\phi(c)$ is a central element in A. This is equivalent with $\operatorname{Supp} A \subset V(c)$. We assume that S is chosen in such a way that $\operatorname{Supp} A \subset S \subset V(c)$.

Let $a = c - \lambda$, where $\lambda \in k$. Then $B = A/(\phi(a))$ also satisfies (A1)(A2). Clearly $V(\ker \phi_B) = V(\ker \phi) \cap V(a)$. If we choose a $\Lambda \in \mathfrak{t}^*/S$ which lies in $V(a)$ then $\mathcal{O}^{(p)}_{\Lambda,B} \subset \mathcal{O}^{(p)}_{\Lambda,A}$ and $\mathcal{O}^{(1)}_{\Lambda,B} = \mathcal{O}^{(1)}_{\Lambda,A}$. Hence the simple objects in $\mathcal{O}^{(p)}_{\Lambda,B}$ and $\mathcal{O}^{(p)}_{\Lambda,A}$ are the same, but the projective objects change.

If $\alpha \in V(\ker \phi_B)$ then it is easy to see that $\beta \underset{\alpha,A}{\Rightarrow} \gamma$ iff $\beta \underset{\alpha,B}{\Rightarrow} \gamma$ and similarly $\beta \underset{A}{\Longleftrightarrow} \gamma$ iff $\beta \underset{B}{\Longleftrightarrow} \gamma$.

Proposition 4.2.1. *Assume that $\Lambda \in \mathfrak{t}^*/S$ lies in $V(a)$ and furthermore that (A3) is satisfied for Λ. Then*

$$H^{(p)}_{\Lambda,B} = H^{(p)}_{\Lambda,A}/(\psi(c)) \tag{4.1}$$

where $\psi : D \to H^{(p)}_{\Lambda,A}$ is the map given by (3.7). In (4.1) it is permissible to put $p = \infty$.

Proof. The case $p = \infty$ follows by taking direct limits, so we assume that p is finite.

By (3.5) $H^{(p)}_{\Lambda,B}$ has the form

$$\begin{pmatrix} \bar{A}_{\alpha_1,\alpha_1}/(a)\bar{A}_{\alpha_1,\alpha_1} & \bar{A}_{\alpha_1,\alpha_2}/(a)\bar{A}_{\alpha_1,\alpha_2} & \cdots \\ \bar{A}_{\alpha_2,\alpha_1}/(a)\bar{A}_{\alpha_2,\alpha_1} & \bar{A}_{\alpha_2,\alpha_2}/(a)\bar{A}_{\alpha_2,\alpha_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where $\bar{A}_{\alpha_i,\alpha_j}$ stands for $A_{\alpha_i-\alpha_j}/A_{\alpha_i-\alpha_j}m^p_{\alpha_j}$. It is now sufficient to prove that for $i = 1, \dots, d$ one has $\phi(c - \alpha_i(c)) = \phi(a)$ (cf. the definition of ψ in (3.7)), or equivalently $\alpha_i(c) = \lambda$. This follows immediately from the fact that by hypothesis $\alpha_i \in V(\ker \phi_B) \subset V(a)$. □

4.3. **Subrings.** Assume that $(\mathfrak{t}, \phi, A)$ satisfies (A1)(A2) and let S_B be a subgroup of S_A. Put $B = \bigoplus_{\alpha \in S_B} A_\alpha$.

Proposition 4.3.1. *Let $\Lambda \in \mathfrak{t}^*/S_A$, $\Gamma \in \mathfrak{t}^*/S_B$, $\Gamma \subset \Lambda$ and $\alpha, \beta, \gamma \in \Gamma$. Then*

(1) *One has $\beta \underset{\alpha,A}{\Rightarrow} \gamma$ iff $\beta \underset{\alpha,B}{\Rightarrow} \gamma$.*

(2) *Similarly $\beta \underset{A}{\Longleftrightarrow} \gamma$ iff $\beta \underset{B}{\Longleftrightarrow} \gamma$.*

(3) *Assume that* Λ *satisfies (A3), with simple modules* $L(\alpha_1)_A, \dots, L(\alpha_d)_A$. *We number the* α_i *so that* $\langle\alpha_i\rangle \cap \Gamma = \emptyset$ *iff* $i > u$ *and choose* $\beta_i \in \langle\alpha_i\rangle \cap \Gamma$, $1 \le i \le u$. *Then the simple objects in* $\mathcal{O}^{(1)}_\Gamma$ *have the form* $L(\beta_i)_B$, $1 \le i \le u$. *Also* $\operatorname{Supp} L(\beta_i)_B = \operatorname{Supp} L(\alpha_i)_A \cap \Gamma$.

Let $e^{(p)}_{\Lambda,\Gamma} \in H^{(p)}_\Lambda$ *be the projection map*

$$\bigoplus_{i=1}^{d} M^{(p)}(\alpha_i)_A \to \bigoplus_{j=1}^{u} M^{(p)}(\beta_j)_B$$

Then

$$H^{(p)}_\Gamma = e^{(p)}_{\Lambda,\Gamma} H^{(p)}_\Lambda e^{(p)}_{\Lambda,\Gamma} \tag{4.2}$$

Furthermore if we put $e^{(\infty)}_{\Lambda,\Gamma} = \varprojlim_p e^{(p)}_{\Lambda,\Gamma}$ *then* (4.2) *holds with* $p = \infty$.

Proof. (1)(2) and the first part of (3) are clear, so we prove the second part of (3). It is also clear that we may assume p finite.

One uses the following fact

$$\operatorname{Hom}_B(M^{(p)}(\alpha)_B, M^{(p)}(\beta)_B) = \operatorname{Hom}_A(M^{(p)}(\alpha)_A, M^{(p)}(\beta)_A) \tag{4.3}$$

Using remark 3.5.3 we may assume $\alpha_1, \dots, \alpha_u \in \Gamma$. Then $H^{(p)}_\Lambda = \operatorname{End}_A(\oplus_{i=1}^d M(\alpha_i)_A)$ and $H^{(p)}_\Gamma = \operatorname{End}_B(\oplus_{i=1}^u M(\alpha_i)_B)$. Then (4.2) follows from (4.3). □

4.4. **Morita equivalence and the → relation.** Now we discuss a construction which is a combination of §4.2 and §4.3. Assume that $(\mathfrak{t}, \phi, A)$ satisfy (A1)(A2). Assume $\mathfrak{g} \subset \mathfrak{t}$ is a subspace. For $\chi \in \mathfrak{g}^*$ we put

$$B^\chi = A^{\mathfrak{g}}/(\mathfrak{g} - \chi(\mathfrak{g}))$$

where

$$\begin{aligned} A^{\mathfrak{g}} &= \{a \in A \mid \forall \pi \in \mathfrak{g} : [\phi(\pi), a] = 0\} \\ &= \bigoplus_{\alpha \in V(\mathfrak{g})} A_\alpha \end{aligned}$$

and

$$\mathfrak{g} - \chi(\mathfrak{g}) = \{\pi - \chi(\pi) \mid \pi \in \mathfrak{g}\} \tag{4.4}$$

Note that $\mathfrak{g} - \chi(\mathfrak{g})$ is contained in the center of $A^{\mathfrak{g}}$.

Let $S_B = \{\alpha \in S_A \mid \alpha(\mathfrak{g}) = 0\}$. By combining the results of §4.2,§4.3 we immediately have the following

Proposition 4.4.1. (1) *Let* $\alpha, \beta \in V(\mathfrak{g} - \chi(\mathfrak{g}))$. *Then* $\alpha \underset{A}{\Longleftrightarrow} \beta$ *iff* $\alpha \underset{B^\chi}{\Longleftrightarrow} \beta$.

(2) *Let* $\Lambda \in \mathfrak{t}^*/S_A$, $\Gamma \in \mathfrak{t}^*/S_B$, $\Gamma \subset \Lambda \cap V(\mathfrak{g} - \chi(\mathfrak{g}))$ *and assume that* Λ *satisfies (A3). Then*

$$H^{(p)}_\Gamma = (e^{(p)}_{\Lambda,\Gamma} H^{(p)}_\Lambda e^{(p)}_{\Lambda,\Gamma})/(\psi(\mathfrak{g}))$$

where $e^{(p)}_{\Lambda,\Gamma}$ *is as in Proposition 4.3.1(3). Furthermore, as in that proposition, it is permissible to put* $p = \infty$.

The various B^χ are related by a Morita context, see [15, §3.6] for background. If $\chi,\chi'\in\mathfrak{g}^*$ then we put

$$B^{\chi,\chi'}=A^{\mathfrak{g}}_{\chi-\chi'}/(\mathfrak{g}-\chi(\mathfrak{g}))A^{\mathfrak{g}}_{\chi-\chi'}$$

where

$$\begin{aligned}A^{\mathfrak{g}}_{\chi-\chi'}&=\{a\in A\mid \forall\pi\in\mathfrak{g}:[\phi(\pi),a]=(\chi-\chi')(\pi)(a)\}\\&=\bigoplus_{\alpha\in V(\mathfrak{g}-(\chi-\chi')(\mathfrak{g}))}A_\alpha\end{aligned}$$

It is clear that $B^{\chi,\chi'}$ is a $B^\chi-B^{\chi'}$-bimodule. Furthermore the multiplication on A defines a Morita context of the form

$$\begin{pmatrix}B^\chi & B^{\chi,\chi'}\\ B^{\chi',\chi} & B^{\chi'}\end{pmatrix}\tag{4.5}$$

We say that χ and χ' are comparable if in (4.5) one has $B^{\chi,\chi'}B^{\chi',\chi}\neq 0$.

Proposition 4.4.2. *If all B^χ are prime then comparability is an equivalence relation on $\mathfrak{g}^*$.*

Proof. **Reflexivity** This is clear.
Symmetry Assume that $B^{\chi,\chi'}B^{\chi',\chi}\neq 0$. Then by (semi)primeness

$$B^{\chi,\chi'}B^{\chi',\chi}B^{\chi,\chi'}B^{\chi',\chi}\neq 0$$

This implies $B^{\chi',\chi}B^{\chi,\chi'}\neq 0$. So symmetry holds.
Transitivity To prove transitivity we use the "triple Morita context"

$$\begin{pmatrix}B^\chi & B^{\chi,\chi'} & B^{\chi,\chi''}\\ B^{\chi',\chi} & B^{\chi'} & B^{\chi',\chi''}\\ B^{\chi'',\chi} & B^{\chi'',\chi'} & B^{\chi''}\end{pmatrix}\tag{4.6}$$

Assume $B^{\chi,\chi'}B^{\chi',\chi}\neq 0$, $B^{\chi',\chi''}B^{\chi'',\chi'}\neq 0$. We claim that

$$B^{\chi,\chi'}B^{\chi',\chi''}B^{\chi'',\chi'}B^{\chi',\chi}\neq 0\tag{4.7}$$

Suppose on the contrary that the left-hand side of (4.7) yields zero. Then

$$B^{\chi',\chi}B^{\chi,\chi'}B^{\chi',\chi''}B^{\chi'',\chi'}B^{\chi',\chi}B^{\chi,\chi'}=0$$

Now by symmetry of comparability, $B^{\chi',\chi}B^{\chi,\chi'}$ is a non-zero ideal in $B^{\chi'}$. Furthermore $B^{\chi',\chi''}B^{\chi'',\chi'}$ is by hypotheses a non-zero ideal in $B^{\chi'}$. Then primeness of $B^{\chi'}$ yields a contradiction.

Hence (4.7) holds. Since we have $B^{\chi,\chi'}B^{\chi',\chi''}\subset B^{\chi,\chi''}$ and $B^{\chi'',\chi'}B^{\chi',\chi}\subset B^{\chi'',\chi}$ we obtain $B^{\chi,\chi''}B^{\chi'',\chi}\neq 0$ which is what we had to prove. □

Remark 4.4.3. If B^χ, $B^{\chi'}$ are prime and χ is comparable to χ' then (4.5) is a so-called "prime Morita context" (see [15, §3.6]). This implies that various properties of B^χ and $B^{\chi'}$ are related. In particular the quotient rings of B^χ and $B^{\chi'}$ (if they exist) are Morita equivalent.

If in the Morita context (4.5) we have that $B^{\chi',\chi}B^{\chi,\chi'}=B^{\chi'}$ then we will write $\chi\to\chi'$. An argument as in the proof of Proposition 4.4.2 shows that this is a transitive relation. Furthermore if $\chi\to\chi'$ and $\chi'\to\chi$ then B^χ and $B^{\chi'}$ are Morita equivalent.

Theorem 4.4.4. *One has $\chi \to \chi'$ iff for all $\alpha \in V(\ker\phi)$ one has that $\langle\alpha\rangle_A \cap V(\mathfrak{g}-\chi'(\mathfrak{g})) \neq \emptyset$ implies $\langle\alpha\rangle_A \cap V(\mathfrak{g}-\chi(\mathfrak{g})) \neq \emptyset$.*

Proof. The proof consists of a chain of equivalences

$$\begin{aligned}
& \chi \to \chi' \\
\text{iff} \quad & \sum_{\gamma\in V(\mathfrak{g}-(\chi-\chi')(\mathfrak{g}))} A_{-\gamma}A_\gamma + (\mathfrak{g}-\chi'(\mathfrak{g}))A_0 = A_0 \\
\text{iff} \quad & \forall\alpha \in V(\mathfrak{g}-\chi'(\mathfrak{g})) \cap V(\ker\phi) : \exists\gamma \in V(\mathfrak{g}-(\chi-\chi')(\mathfrak{g})) : A_{-\gamma}A_\gamma \not\subset Am_\alpha \\
\text{iff} \quad & \forall\alpha \in V(\mathfrak{g}-\chi'(\mathfrak{g})) \cap V(\ker\phi) : \exists\gamma \in V(\mathfrak{g}-(\chi-\chi')(\mathfrak{g})) : \alpha+\gamma \in V(\ker\phi) \\
& \qquad\qquad \text{and } \alpha+\gamma \underset{A}{\Longleftrightarrow} \alpha \\
\text{iff} \quad & \forall\alpha \in V(\ker\phi) : \langle\alpha\rangle_A \cap V(\mathfrak{g}-\chi'(\mathfrak{g})) \neq \emptyset \Rightarrow \langle\alpha\rangle_A \cap V(\mathfrak{g}-\chi(\mathfrak{g})) \neq \emptyset
\end{aligned}$$

The third equivalence follows from lemma 3.1.9(6). □

If J is an ideal in B^χ then we set

$$\tilde{J} = \{x \in B^{\chi'} \mid B^{\chi,\chi'} x B^{\chi',\chi} \subset J\}$$

Clearly $\tilde{J} = B^{\chi'}$ iff $B^{\chi,\chi'}B^{\chi',\chi} \subset J$. Also by [15, Thm 3.6.2, Prop. 3.6.5(ii)] $J \mapsto \tilde{J}$ yields a 1-1, order preserving correspondence between the primitive ideals of B^χ not containing $B^{\chi,\chi'}B^{\chi',\chi}$ and those of $B^{\chi'}$ not containing $B^{\chi',\chi}B^{\chi,\chi'}$.

If $M \in \mathcal{O}_A^{(1)}$ then let us define M^χ by $\oplus_{\alpha\in V(\mathfrak{g}-\chi(\mathfrak{g}))} M_\alpha$. This is a B^χ-module which is simple if M is simple. Furthermore $\begin{pmatrix} M^\chi \\ M^{\chi'} \end{pmatrix}$ is a left module over (4.5) and if M is simple and $M^\chi, M^{\chi'} \neq 0$ then $B^{\chi',\chi}M^\chi = M^{\chi'}$, $B^{\chi,\chi'}M^{\chi'} = M^\chi$. This allows us to prove the following result.

Proposition 4.4.5. *Let $\alpha \in V(\mathfrak{g}-\chi(\mathfrak{g}))$. Then*

$$\widetilde{J(\alpha)}_{B^\chi} = \begin{cases} J(\beta)_{B^{\chi'}} & \text{if } \beta \in V(\mathfrak{g}-\chi'(\mathfrak{g})) \cap \langle\alpha\rangle_A \\ B^{\chi'} & \text{if } V(\mathfrak{g}-\chi'(\mathfrak{g})) \cap \langle\alpha\rangle_A = \emptyset \end{cases}$$

Proof. We have

$$L(\alpha)_A^{\chi'} = \begin{cases} L(\beta)_{B^{\chi'}} & \text{if } \beta \in V(\mathfrak{g}-\chi'(\mathfrak{g})) \cap \langle\alpha\rangle_A \\ 0 & \text{if } V(\mathfrak{g}-\chi'(\mathfrak{g})) \cap \langle\alpha\rangle_A = \emptyset \end{cases}$$

To simplify the notations we put $J = J(\alpha)_{B^\chi}$, $L = L(\alpha)_A^\chi = L(\alpha)_{B^\chi}$ and $J' = J(\beta)_{B^{\chi'}}$, $L' = L(\beta)_{B^{\chi'}}$ (if the latter two are defined).

Case 1. $V(\mathfrak{g}-\chi'(\mathfrak{g})) \cap \langle\alpha\rangle_A \neq \emptyset$. Then $\begin{pmatrix} L \\ L' \end{pmatrix}$ is a module over (4.5). Clearly $B^{\chi,\chi'}J'B^{\chi',\chi}L = 0$ and hence $J' \subset \tilde{J}$.

To prove the opposite inclusion we note that $B^{\chi,\chi'}\tilde{J}B^{\chi',\chi}L = 0$. Since $B^{\chi',\chi}L = L'$, $B^{\chi,\chi'}L' = L$ we must necessarily have $\tilde{J}L' = 0$ which is equivalent to $\tilde{J} \subset J'$.

Case 2. $V(\mathfrak{g}-\chi'(\mathfrak{g})) \cap \langle\alpha\rangle_A = \emptyset$. Now $\begin{pmatrix} L \\ 0 \end{pmatrix}$ is a module over (4.5). Hence $B^{\chi,\chi'}B^{\chi',\chi}L = 0$ and thus $\tilde{J} = B^{\chi'}$. □

4.5. **Some quotients by two sided ideals.** Let $(\mathfrak{t}, \phi, A)$ be as before. We prove the following result which will be used afterwards.

Proposition 4.5.1. *Assume that $J \subset A$ is a two sided ideal with the following properties*

(1) $\forall \alpha \in \mathfrak{t}^* : J_\alpha = J_0 A_\alpha + A_\alpha J_0$
(2) *There exists a subspace $\mathfrak{g} \subset \mathfrak{t}$ and $\chi_1, \dots, \chi_p \in \mathfrak{g}$ such that*
$$J_0 = \phi((\mathfrak{g} - \chi_1(\mathfrak{g})) \cap \dots \cap (\mathfrak{g} - \chi_p(\mathfrak{g})))$$
where for $S \subset A$, (S) denotes the ideal generated by S.

Then there is an isomorphism between A/J and
$$\begin{pmatrix} B^{\chi_1} & B^{\chi_1,\chi_2} & & \\ B^{\chi_2,\chi_1} & B^{\chi_2} & & \\ & & \ddots & \\ & & & B^{\chi_p} \end{pmatrix}$$
where B^{χ_i,χ_j} is as defined in §4.4.

Proof. Let $(e_i)_{i=1,\dots,p} \in D$ be representatives for a maximal set of orthogonal idempotents in $D/(\mathfrak{g} - \chi_1(\mathfrak{g})) \cap \dots \cap (\mathfrak{g} - \chi_p(\mathfrak{g}))$. Thus e_i, as a function on $\mathfrak{t}^*$ has the property that $e_i \mid V(\mathfrak{g} - \chi_j(\mathfrak{g})) = \delta_{ij}$.

To make the next computation we choose a basis $(\pi_i)_{i=1,\dots,n}$ for $\mathfrak{t}$ and we use this basis to identify $\mathfrak{t}^*$ with k^n. Then e_i is a polynomial $e_i(\pi_1, \dots \pi_n)$. Let $a \in A_\alpha$ where $\alpha \in V(\mathfrak{g} - (\chi_i - \chi_j)(\mathfrak{g}))$. Then
$$e_i a e_j = a e_i(\pi_1 + \alpha_1, \dots, \pi_n + \alpha_n) e_j$$
Now we claim that
$$e_i(\pi_1 + \alpha_1, \dots, \pi_n + \alpha_n) e_j \cong e_j \mod \phi^{-1}(J_0)$$
To see this one has to show that for $k = 1, \dots, p$
$$e_i(\pi_1 + \alpha_1, \dots, \pi_n + \alpha_n) e_j \mid V(\mathfrak{g} - \chi_k(\mathfrak{g})) = \delta_{jk}$$
If $j \neq k$ then this is clear and for $j = k$, it follows from $e_i \mid V(\mathfrak{g} - \chi_i(\mathfrak{g})) = 1$.

Thus we have shown that in A/J one has for $a \in A^{\mathfrak{g}}_{\chi_i - \chi_j}$
$$\overline{ae_j} = \overline{e_i a e_j} = \overline{e_i a} \tag{4.8}$$
(the last equality follows by symmetry).

Now let $\epsilon_{ij} : A^{\mathfrak{g}}_{\chi_i - \chi_j} \to \bar{e}_i(A/J)\bar{e}_j$ be defined by $a \mapsto \bar{e}_i \bar{a} \bar{e}_j$. Then (4.8) implies that
$$\epsilon_{ij}(a)\epsilon_{jk}(b) = \epsilon_{ik}(ab) \tag{4.9}$$
Let $\alpha \in \mathfrak{t}^*$. We will analyze $\bar{e}_i(A/J)_\alpha \bar{e}_j$ more closely. We have $(A/J)_\alpha = A_\alpha / I A_\alpha$ where
$$I = [(\mathfrak{g} - (\chi_1 + \alpha|_{\mathfrak{g}})(\mathfrak{g})) \cap \dots \cap (\mathfrak{g} - (\chi_p + \alpha|_{\mathfrak{g}})(\mathfrak{g}))] + [(\mathfrak{g} - \chi_1(\mathfrak{g})) \cap \dots \cap (\mathfrak{g} - \chi_p(\mathfrak{g}))]$$
Hence
$$\bar{e}_i(A_\alpha / I A_\alpha) = \begin{cases} A_\alpha / (\mathfrak{g} - \chi_i(\mathfrak{g})) A_\alpha & \text{if } \chi_i - \alpha \mid_{\mathfrak{g}} \in \{\chi_1, \dots, \chi_p\} \\ 0 & \text{otherwise} \end{cases}$$
So assume $\chi_i - \alpha|_{\mathfrak{g}} \in \{\chi_1, \dots, \chi_p\}$. Then
$$\bar{e}_i(A_\alpha / I A_\alpha)\bar{e}_j = A_\alpha / (\mathfrak{g} - \chi_i(\mathfrak{g})) A_\alpha \bar{e}_j = A_\alpha / A_\alpha(\mathfrak{g} - (\chi_i - \alpha|_{\mathfrak{g}})(\mathfrak{g}))\bar{e}_j$$

which yields

$$\bar{e}_i(A/J)_\alpha \bar{e}_j = \begin{cases} A_\alpha/(\mathfrak{g} - \chi_i(\mathfrak{g}))A_\alpha & \text{if } \alpha|_{\mathfrak{g}} = \chi_i - \chi_j \\ 0 & \text{otherwise} \end{cases}$$

So ϵ_{ij} is surjective, and the kernel is equal to $(\mathfrak{g} - \chi_i(\mathfrak{g})) \mid A^{\mathfrak{g}}_{\chi_i - \chi_j}$. In other words, ϵ_{ij} defines an isomorphism between B^{χ_i,χ_j} and $\bar{e}_i(A/J)\bar{e}_j$. This together with (4.9) proves the proposition. □

5. The algebras introduced by S.P. Smith

The machinery introduced in §3 is geared towards the study of rings of differential operators on toric varieties and quotients under torus actions. However there are many more examples. A non-trivial example is given by the analogues of $U(\mathfrak{s}l_2)$ introduced by S.P. Smith in [23]. These are defined as follows. Let $A = k[H, E, F]$ where

$$[H, E] = E, \qquad [H, F] = -F, \qquad [E, F] = f(H)$$

where f is a fixed polynomial in one variable.

According to [23, Prop. 1.5], the center of A is generated by the "Casimir element"

$$\Omega = EF + FE + \frac{1}{2}(u(H + 1) + u(H))$$

where $u \in k[x]$ is such that

$$\frac{1}{2}(u(x + 1) - u(x)) = f(x) \tag{5.1}$$

If we put $\mathfrak{t} = kH + k\Omega$ and $D = k[H, \Omega]$ then $\mathfrak{t}$ acts semi-simply on A, with weight space decomposition

$$A = \cdots \oplus DF^2 \oplus DF \oplus D \oplus DE \oplus DE^2 \oplus \cdots$$

Using the material in §3 one can now recover, without too much work, most of the results in [23]. Of course this will not be our aim below. Instead we hope to make clear that a systematic study of the $\Longleftrightarrow$ relation makes possible a unified treatment of otherwise disparate results. In particular we give a new proof of a result by Bavula [3] and Hodges [6] which computes the global dimension of A. Finally we also give a description of the category of finite dimensional representations of A. We believe this result is new.

Throughout we identify $\mathfrak{t} = kH \oplus k\Omega$ and its dual $\mathfrak{t}^*$ with k^2 in the natural way. Thus an element $\alpha \in \mathfrak{t}^*$ will be written as (α_1, α_2) with $\alpha_1, \alpha_2 \in k$.

5.1. **The $\Longleftrightarrow$ relation.** The following identities are easily proved by induction.

$$\begin{aligned} EF^n &= \frac{1}{2}F^{n-1}(\Omega - u(H - n + 1)), \qquad \text{for } n \geq 1 \\ FE^n &= \frac{1}{2}E^{n-1}(\Omega - u(H + n)), \qquad \text{for } n \geq 1 \end{aligned} \tag{5.2}$$

Fix $\alpha \in \mathfrak{t}^*$. Using (5.2) we may now describe the $\underset{\alpha}{\Rightarrow}$-relation (which was defined just before lemma 3.1.9). To simplify the notation we write $\alpha + n$ for $(\alpha_1 + n, \alpha_2)$ if $n \in \mathbb{Z}$.

All basic instances of the $\underset{\alpha}{\Rightarrow}$-relation are described by the following four cases : for $n \geq 1$

(1) $\alpha+n \underset{\alpha}{\Rightarrow} \alpha+n-1$ iff $\alpha_2-u(\alpha_1+n)\neq 0$;
(2) $\alpha-n \underset{\alpha}{\Rightarrow} \alpha-n+1$ iff $\alpha_2-u(\alpha_1-n+1)\neq 0$;

and for $n\geq 0$

(3) $\alpha+n \underset{\alpha}{\Rightarrow} \alpha+n+1$;
(4) $\alpha-n \underset{\alpha}{\Rightarrow} \alpha-n-1$.

We denote by $r_1,\dots,r_t$ the roots (without repetition) of $\alpha_2-u(x)$ that are congruent to $\alpha_1 \mod \mathbb{Z}$, ordered in ascending order (this makes sense!).

Let $i\in\{0,\dots,t\}$ be such that $r_i\leq\alpha_1<r_{i+1}$, where for convenience we assume that $r_0=\alpha_1-\infty$, $r_{t+1}=\alpha_1+\infty$.

Then (1)(2)(3)(4) may be translated as follows : for $\beta,\gamma\in\mathfrak{t}^*$: $\beta \underset{\alpha}{\Rightarrow} \gamma$ iff $\alpha_2=\beta_2=\gamma_2$, $\alpha_1\cong\beta_1\cong\gamma_1 \mod \mathbb{Z}$ and

(1) For $j\leq i: r_{j-1}\leq\beta_1<r_j\Rightarrow\gamma_1<r_j$.
(2) For $j\geq i+2: r_{j-1}\leq\beta_1<r_j\Rightarrow\gamma_1\geq r_{j-1}$

We deduce that the equivalence class for $\iff$ of α is given by

$$\langle\alpha\rangle=\{\beta\in\mathfrak{t}^*\mid\beta_2=\alpha_2,\beta_1\cong\alpha_1 \mod \mathbb{Z}, r_i\leq\beta_1<r_{i+1}\} \tag{5.3}$$

and thus $\overline{\langle\alpha\rangle}=k\times\{\alpha_2\}$ iff $i=0,t$. In the other cases $\overline{\langle\alpha\rangle}$ equals $\langle\alpha\rangle$, which is of the form "finite set" $\times\{\alpha_2\}$

5.2. **The category $\mathcal{O}^{(\infty)}$.** To study modules over A we now compute $H_\Lambda^{(\infty)}$ where $\Lambda=\alpha+\operatorname{Supp}A=\{\beta\mid\beta_2=\alpha_2,\beta_1\cong\alpha_1 \mod \mathbb{Z}\}$.

We recall that $H_\Lambda^{(\infty)}$ is the completion of H_Λ at the ideal (H,Ω) where H_Λ is defined by (3.8).

We choose $\epsilon_0<\epsilon_1<\cdots<\epsilon_t$ in such a way that $(\epsilon_i,\alpha_2), i=0,\dots,t$ are representatives for the equivalence classes of $\iff$ in Λ and we put $\delta_i=\epsilon_i-\epsilon_{i-1}\in\mathbb{Z}$, $i=1,\dots,t$. Then H_Λ is given by

$$\begin{pmatrix} D & DF^{\delta_1} & DF^{\delta_1+\delta_2} & \cdots\\ DE^{\delta_1} & D & DF^{\delta_2} & \cdots\\ DE^{\delta_1+\delta_2} & DE^{\delta_2} & D & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}$$

and the map $\psi: D\to H_\Lambda$ is given by

$$H\mapsto\begin{pmatrix} H-\epsilon_0 & & \\ & \ddots & \\ & & H-\epsilon_t\end{pmatrix}$$
$$\Omega\mapsto\begin{pmatrix} \Omega-\alpha_2 & & \\ & \ddots & \\ & & \Omega-\alpha_2\end{pmatrix}$$

We consider the quiver

$$
\begin{tikzcd}
\bullet \arrow[loop above, "H_0"] \arrow[loop below, "\Omega_0"] \arrow[r, shift left, "X_1"] & \bullet \arrow[l, shift left, "Y_1"] \arrow[loop above, "H_1"] \arrow[loop below, "\Omega_1"] \arrow[r, shift left, "X_2"] & \bullet \arrow[l, shift left, "Y_2"] \arrow[loop above, "H_2"] \arrow[loop below, "\Omega_2"] & \cdots & \bullet \arrow[loop above, "H_{t-1}"] \arrow[loop below, "\Omega_{t-1}"] \arrow[r, shift left, "X_t"] & \bullet \arrow[l, shift left, "Y_t"] \arrow[loop above, "H_t"] \arrow[loop below, "\Omega_t"]
\end{tikzcd} \tag{5.4}
$$

subject to the relations

$$X_iY_i = \left(\frac{1}{2}\right)^{\delta_i} \prod_{j=0}^{\delta_i-1} (\Omega_i + \alpha_2 - u(H_i + \epsilon_i - j) \tag{5.5}$$

$$Y_iX_i = \left(\frac{1}{2}\right)^{\delta_i} \prod_{j=1}^{\delta_i} (\Omega_{i-1} + \alpha_2 - u(H_{i-1} + \epsilon_{i-1} + j) \tag{5.6}$$

$$H_iX_i = X_iH_{i-1} \tag{5.7}$$

$$H_{i-1}Y_i = Y_iH_i \tag{5.8}$$

$$\Omega_iX_i = X_i\Omega_{i-1} \tag{5.9}$$

$$\Omega_{i-1}Y_i = Y_i\Omega_i \tag{5.10}$$

(we have used the convention that a path $\xrightarrow{a}\xrightarrow{b}$ is written as ba.)

Then there is an isomorphism from the path algebra of this quiver to H_Λ sending H_i (resp. Ω_i) to the diagonal matrix with $H - \epsilon_i$ (resp. $\Omega - \alpha_2$) in the $(i+1)$'st position and zeroes elsewhere, and X_i (resp. Y_i) to the matrix with i'th entry E^{δ_i} (resp. F^{δ_i}) on the subdiagonal (resp. super-diagonal) and zeroes elsewhere.

For example when $t = 1$ the isomorphism is given by

$$
\begin{aligned}
H_0 &\to \operatorname{diag}(H - \epsilon_0, 0), & H_1 &\to \operatorname{diag}(0, H - \epsilon_1) \\
\Omega_0 &\to \operatorname{diag}(\Omega - \alpha_2, 0), & \Omega_1 &\to \operatorname{diag}(0, \Omega - \alpha_2) \\
X_1 &\to \left(\begin{smallmatrix} 0 & 0 \\ E^{\delta_1} & 0 \end{smallmatrix}\right), & Y_1 &\to \left(\begin{smallmatrix} 0 & F^{\delta_1} \\ 0 & 0 \end{smallmatrix}\right)
\end{aligned}
$$

The fact that (5.5) transforms to an identical relation follows from the identity

$$E^\delta F^\delta = \left(\frac{1}{2}\right)^{\delta} \prod_{j=0}^{\delta-1} (\Omega - u(H - j))$$

in A (see [23, Appendix]). Similarly for relations (5.6)...(5.10). It is now easy to check that this map is an isomorphism and we identify H_Λ with this path algebra. We now have $\psi(H) = \sum_{i=0}^t H_i$, $\psi(\Omega) = \sum_{i=0}^t \Omega_i$.

Now we make some simplifications. We want to complete H_Λ at the ideal $(\psi(H), \psi(\Omega))$, and hence every factor in (5.5)(5.6) that is not zero if we put $\psi(H) = 0, \psi(\Omega) = 0$ will become a unit. Using the special nature of the equations (5.5)...(5.10) the reader may verify that one may eliminate these units by changing the variables X_i, Y_i. Hence we have almost proved the following.

Theorem 5.2.1. *$H_\Lambda^{(\infty)}$ is isomorphic to the completed path algebra of the quiver* (5.4) *subject to the relations* (5.7)...(5.10) *and the relations*

$$X_iY_i = \Omega_i + \alpha_2 - u(H_i + r_i) \tag{5.11}$$

$$Y_iX_i = \Omega_{i-1} + \alpha_2 - u(H_{i-1} + r_i) \tag{5.12}$$

In particular the full subcategory of $\mathcal{O}_\Lambda^{(\infty)}$, consisting of finitely generated objects, is equivalent with the category of finite dimensional representations of the quiver (5.4) *(subject to the relations* (5.7)...(5.12)*), having the additional property that sufficiently long paths act as zero.*

Proof. Everything is clear, except perhaps the fact that completing the path algebra at $(\psi(H), \psi(\Omega))$ is the same as completing at the ideal generated by the paths of length one. To prove this one has to show that the two corresponding adic filtrations are cofinal. This is left as an exercise. □

Remark 5.2.2. We prefer the relations given by the above theorem since they make clear the role of both H and Ω. However it is possible to eliminate Ω. Then (5.4) is replaced by the quiver

$$\begin{tikzcd} \bullet \arrow[loop above, "H_0"] \arrow[r, shift left, "X_1"] & \bullet \arrow[loop above, "H_1"] \arrow[l, shift left, "Y_1"] \arrow[r, shift left, "X_2"] & \bullet \arrow[loop above, "H_2"] \arrow[l, shift left, "Y_2"] & \cdots & \bullet \arrow[loop above, "H_{t-1}"] \arrow[r, shift left, "X_t"] & \bullet \arrow[loop above, "H_t"] \arrow[l, shift left, "Y_t"] \end{tikzcd} \tag{5.13}$$

subject to the relations

$$X_iY_i - Y_{i+1}X_{i+1} = u(H_i + r_{i+1}) - u(H_i + r_i), \text{ for } i = 1, \dots, t-1 \tag{5.14}$$

$$H_iX_i = X_iH_{i-1} \tag{5.15}$$

$$H_{i-1}Y_i = Y_iH_i \tag{5.16}$$

Now we draw some conclusions.

5.3. **Smith's $\mathcal{O}$-category.** Let us recall Smith's definition of $\mathcal{O}$ (which is a direct generalization of [5]). An A-module is in $\mathcal{O}$ iff

(1) M is the sum of its H-weight spaces.
(2) For all $m \in M$, $\dim(k[E] \cdot m) < \infty$.
(3) M is a finitely generated A-module.

If $\Lambda \in \mathfrak{t}^*/\operatorname{Supp} A$ is as above then $\mathcal{O}_\Lambda$ is defined as the full subcategory of $\mathcal{O}$ of those objects having their weights in Λ. It is clear that $\mathcal{O} \subset \mathcal{O}^{(\infty)}$ and $\mathcal{O}_\Lambda \subset \mathcal{O}_\Lambda^{(\infty)}$. Furthermore if $M \in \mathcal{O}_\Lambda^{(\infty)}$ then (1)(2)(3) are equivalent with

(1') $\forall \beta \in \Lambda : (H - \beta_1)M_\beta = 0$;
(2') M does not have $L(\epsilon_t)$ as a subquotient;
(3') M has finite length.

We then easily prove the following (this has also been observed in [33]).

Proposition 5.3.1. *The category $\mathcal{O}_\Lambda$ is equivalent with the category of finite dimensional representations over the quiver*

$$\begin{tikzcd} \bullet \arrow[r, shift left, "X_1"] & \bullet \arrow[l, shift left, "Y_1"] \arrow[r, shift left, "X_2"] & \bullet \arrow[l, shift left, "Y_2"] & \cdots & \bullet \arrow[r, shift left, "X_{t-1}"] & \bullet \arrow[l, shift left, "Y_{t-1}"] \end{tikzcd} \tag{5.17}$$

with relations

$$\begin{aligned} X_iY_i - Y_{i+1}X_{i+1} &= 0, \qquad \text{for } i = 1, \ldots, t-2 \\ X_{t-1}Y_{t-1} &= 0 \end{aligned}$$

Proof. By using (3.10), and by tracing back the computations in §5.2 we see that objects in $\mathcal{O}_\Lambda$ correspond to finite dimensional representations of the quiver (5.4) with dimension vector $(d_0, \ldots, d_{t-1}, 0)$, having the property that the $(H_i)_i$ act as zero.

By using the fact that by definition $\alpha_2 = u(r_i)$ we obtain (5.17). Note that long paths are automatically zero in (5.17). □

5.4. **Finite dimensional representations.** We have noted in §3.1 that the category of finite dimensional representations lies in in $\mathcal{O}^{(\infty)}$. The following proposition describes this subcategory.

Proposition 5.4.1. *The quiver*

$$\begin{tikzcd} \bullet \arrow[loop above, "H_1"] \arrow[r, shift left, "X_2"] & \bullet \arrow[loop above, "H_2"] \arrow[l, shift left, "Y_2"] \arrow[r, shift left, "X_3"] & \bullet \arrow[loop above, "H_3"] \arrow[l, shift left, "Y_3"] & \cdots & \bullet \arrow[loop above, "H_{t-2}"] \arrow[r, shift left, "X_{t-1}"] & \bullet \arrow[loop above, "H_{t-1}"] \arrow[l, shift left, "Y_{t-1}"] \end{tikzcd} \tag{5.18}$$

with relations :
if $t \geq 3$

$$\begin{aligned} -Y_2X_2 &= u(H_1 + r_2) - u(H_1 + r_1) \\ X_iY_i - Y_{i+1}X_{i+1} &= u(H_i + r_{i+1}) - u(H_i + r_i), \quad \text{for } i = 2, \ldots, t-2 \\ X_{t-1}Y_{t-1} &= u(H_{t-1} + r_t) - u(H_{t-1} + r_{t-1}) \\ H_iX_i &= X_iH_{i-1} \\ H_{i-1}Y_i &= Y_iH_i \end{aligned} \tag{5.19}$$

if $t = 2$

$$0 = u(H_1 + r_2) - u(H_1 + r_1) \tag{5.19bis}$$

has a finite dimensional path algebra, say of dimension N.

Furthermore one has that the category of finite dimensional representations in $\mathcal{O}_\Lambda^{(\infty)}$ is non-trivial iff $t \geq 2$, and is equivalent with the category of finite dimensional representations over the quiver satisfying the relations (5.19(bis)) *together with the relations*

$$H_i^N = 0, \qquad \text{for } i = 1, \ldots, t-1 \tag{5.20}$$

Proof. The proof that (5.18) has a finite dimensional path algebra is an exercise which is left to the reader (see example 5.4.4 below for the case $t = 3$).

A finitely generated object in $\mathcal{O}_\Lambda^{(\infty)}$ is finite dimensional if and only if it contains no composition factors of the form $L(\epsilon_0)$, $L(\epsilon_t)$. The proposition now follows directly from Theorem 5.2.1. Note that the completion has been replaced by the equivalent operation of adding the relations (5.20). □

Remark 5.4.2. It was shown in [33] that the category of finite dimensional representations of A, with generalized weights lying in Λ, has projective covers. This is equivalent with saying that it is given by the representations of some finite dimensional algebra. However this algebra was not determined explicitly.

In [23, Cor 3.8] Smith shows that any ideal of finite codimension in $A/(\Omega - \alpha)$ is eventually idempotent. Proposition 5.4.1 allows us to do better.

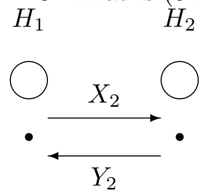
Corollary 5.4.3. *Every ideal in A of finite codimension is eventually idempotent*

Proof. Let $J \subset A$ be an ideal of finite codimension. We will show that the length of A/J as a left A-module is bounded in terms of $|V(J_0)|$. Since $|V(J_0)| = |V((J^2)_0)|$ this proves what we want.

Let's suppose $V(J_0) = \{\beta_1, \dots, \beta_q\}$. Then there exists p such that A/J is a quotient, as left A-module of $M^{(p)}(\beta_1) \oplus \cdots \oplus M^{(p)}(\beta_q)$. Applying $F^{(\infty)}$ shows that $F^{(\infty)}(A/J)$ is a finite dimensional quotient of $H_{\Lambda_1}^{(\infty)} \oplus \cdots \oplus H_{\Lambda_q}^{(\infty)}$, where Λ_i is chosen to contain β_i. According to Proposition 5.4.1 there exists a finite dimensional quotient, say Q_i of $H_{\Lambda_i}^{(\infty)}$ such that finite dimensional objects in $\mathcal{O}_{\Lambda_i}^{(\infty)}$ correspond to finite dimensional representations of Q_i. Furthermore it is easy to see that $\dim Q_i$ may be uniformly bounded in terms of $\deg u(H)$ (see (5.1)), say by M.

Hence one deduces that the length of $F^{(\infty)}(A/J)$ is bounded by qM, and so the same holds for A/J. $\square$

Example 5.4.4. Assume that $t = 3$. That is (5.18) is the quiver



with relations

$$Y_2X_2 = u(H_1 + r_1) - u(H_1 + r_2)$$
$$X_2Y_2 = u(H_2 + r_3) - u(H_2 + r_2)$$

together with the two last equations of (5.19).

Expanding $Y_2X_2Y_2$ in two ways yields

$$(u(H_1 + r_1) - u(H_1 + r_3))Y_2 = 0 \tag{5.21}$$

Similarly by expanding $X_2Y_2X_2$

$$X_2(u(H_1 + r_1) - u(H_1 + r_3)) = 0$$

Multiplying (5.21) on the left and on the right with X_2 yields

$$(u(H_1 + r_1) - u(H_1 + r_3))(u(H_1 + r_1) - u(H_1 + r_2)) = 0$$
$$(u(H_2 + r_1) - u(H_2 + r_3))(u(H_2 + r_3) - u(H_2 + r_2)) = 0$$

Put $\theta_i = u(H + r_i)$. We obtain that the path algebra of (5.18) is given by

$$\begin{pmatrix} k[H]/(\theta_1 - \theta_3)(\theta_1 - \theta_2) & Yk[H]/(\theta_1 - \theta_3) \\ Xk[H]/(\theta_1 - \theta_3) & k[H]/(\theta_1 - \theta_3)(\theta_2 - \theta_3) \end{pmatrix} \tag{5.22}$$

where X, Y commute with H and satisfy the relations

$$\begin{aligned} YX &= \theta_1 - \theta_2 \\ XY &= \theta_3 - \theta_2 \end{aligned} \tag{5.23}$$

Put $n_1 = \operatorname{ord}_{(H)}(\theta_2 - \theta_3)$, $n_2 = \operatorname{ord}_{(H)}(\theta_1 - \theta_3)$ and $n_3 = \operatorname{ord}_{(H)}(\theta_1 - \theta_2)$. Note that two of these numbers have to be equal. Then the completion of (5.22) is given by

$$\begin{pmatrix} k[H]/(H^{n_2+n_3}) & Yk[H]/(H^{n_2}) \\ Xk[H]/(H^{n_2}) & k[H]/(H^{n_1+n_2}) \end{pmatrix}$$

where X, Y still satisfy (5.23).

5.5. **Primitive ideals and primitive quotients.** We start by reproving the following proposition.

Proposition 5.5.1. [23] *Let J be a primitive ideal in A. Then*

(1) *J contains some $\Omega - \lambda$, $\lambda \in k$;*
(2) *J is of the form $\operatorname{Ann}_A L(\alpha)$ where $L(\alpha)$ may be chosen in Smith's $\mathcal{O}$-category;*
(3) *$(\Omega - \lambda)$ is a primitive ideal for all $\lambda \in k$;*
(4) *J is generated by $J \cap k[H, \Omega]$.*

Proof. (1) follows from Quillen's lemma. For $\lambda \in k$ put $B = A/(\Omega - \lambda)$. It follows from (5.3) that the hypotheses for Theorem 3.2.4 are satisfied for B. This implies (2)(4). Furthermore if we choose $\alpha = (\alpha_1, \lambda)$ in such a way that $\overline{\langle \alpha \rangle} = k \times \{\lambda\}$ then $\operatorname{Ann}_B L(\alpha) = 0$. This proves (3). □

Now fix $\lambda \in k$ and put $B = A/(\Omega - \lambda)$ as above. These are the minimal primitive quotients of A. One may prove the following results.

Lemma 5.5.2. [23] *B is a domain.*

Proof. This follows from (5.3) and Proposition 3.4.1. It suffices to choose $\alpha = (\alpha_1, \lambda)$ in such a way that α_1 is "large" in its congruence class $\mod \mathbb{Z}$. □

Proposition 5.5.3. [8, §3] *B is simple if and only if the polynomial $\lambda - u(x)$ has no two distinct roots which differ by an element of $\mathbb{Z}$.*

Proof. This follows from the above lemma, (5.3) and Proposition 3.3.1. □

Proposition 5.5.4. [3][6] *One has the following*

(1) *The global dimension of B is finite if and only if the polynomial $\lambda - u(x)$ has no multiple roots.*
(2) *If $\lambda - u(x)$ has no multiple roots then*

$$\operatorname{gl\,dim} B = \begin{cases} 2 & \text{if } \lambda - u(x) \text{ has two roots which differ by a non-zero element of } \mathbb{Z}. \\ 1 & \text{otherwise} \end{cases}$$

Proof. We first prove (1) and (2) for *graded* global dimension. We apply the criterion given by corollary 3.5.11. Let $\alpha = (\alpha_1, \lambda) \in \mathfrak{t}^*$, $\Lambda = \alpha + \operatorname{Supp} A$. We have to determine when $H^{(\infty)}_{\Lambda,B}$ has finite global dimension.

Using 4.2.1 we see that $H^{(\infty)}_{\Lambda,B} = H^{(\infty)}_{\Lambda,A}/(\psi(\Omega))$. Hence to obtain $H^{(\infty)}_{\Lambda,B}$ we have to set $(\Omega_i)_{i=0,\dots,t} = 0$ in (5.7)…(5.12).

Thus $H^{(\infty)}_{\Lambda,B}$ is the completed path algebra of the quiver

$$\begin{tikzcd}
\bullet \arrow[loop above, "H_0"] \arrow[r, shift left, "X_1"] & \bullet \arrow[loop above, "H_1"] \arrow[l, shift left, "Y_1"] \arrow[r, shift left, "X_2"] & \bullet \arrow[loop above, "H_2"] \arrow[l, shift left, "Y_2"] & \cdots & \bullet \arrow[loop above, "H_{t-1}"] \arrow[r, shift left, "X_t"] & \bullet \arrow[loop above, "H_t"] \arrow[l, shift left, "Y_t"]
\end{tikzcd}$$

with relations

$$X_iY_i = \lambda - u(H_i + r_i) \tag{5.24}$$

$$Y_iX_i = \lambda - u(H_{i-1} + r_i) \tag{5.25}$$

$$H_iX_i = X_iH_{i-1}$$

$$H_{i-1}Y_i = Y_iH_i$$

After dropping unit factors, we may replace (5.24)(5.25) by

$$X_iY_i = H_i^{m_i}$$

$$Y_iX_i = H_{i-1}^{m_i}$$

where m_i is the multiplicity of the root r_i of $\lambda - u(x)$.

The completed path algebra of the resulting quiver is a so called "tiled" order

$$H^{(\infty)}_{\Lambda,B} = \begin{pmatrix} k[[H]] & k[[H]]H^{m_1} & k[[H]]H^{m_1+m_2} & \cdots \\ k[[H]] & k[[H]] & k[[H]]H^{m_2} & \cdots \\ k[[H]] & k[[H]] & k[[H]] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.26}$$

The global dimension of such an order is given by [7, lem. 2.7, cor. 2.10]

$$\operatorname{gl\,dim} H^{(\infty)}_{\Lambda,B} = \begin{cases} \infty & \text{if some } m_i \neq 1 \\ 2 & \text{if } t \geq 2 \text{ and all } m_i = 1 \\ 1 & \text{otherwise} \end{cases}$$

This finishes the proof of (1)(2) for graded global dimension. Then remark 3.5.12 implies that (1) is also true for ordinary global dimension. So we are left with (2). To handle this case we may proceed as in [6].

Assume that $\operatorname{gl\,dim} B < \infty$. One may define a filtration on B such that the injective dimension of $\operatorname{gr} B$ is equal to 2. To do this put $\deg H = 1$, $\deg E = \deg F = N$ where $N \gg 0$. This defines a filtration on A such that $\operatorname{gr} A = k[H, E, F]$, a polynomial ring in three variables. Then for the induced filtration on B one has $\operatorname{gr} B = k[H, E, F]/(EF)$ which clearly has the right injective dimension.

Now we use the well known formulas

$$\operatorname{gl\,dim} B = \operatorname{inj\,dim}_B B = \operatorname{GKdim} B - \min \operatorname{GKdim} M = 2 - \min \operatorname{GKdim} M$$

where the minimum is taken over all finitely generated B-modules (see the proof of Theorem 8.4.1 and lemma 9.1.2). Now one always has $\operatorname{GKdim} L(\alpha) \leq 1$. Hence we obtain that $\operatorname{gl\,dim} B = 2$ if and only if B has a finite dimensional representation. Otherwise $\operatorname{gl\,dim} B = 1$. Now finite dimensional representations are of the form $L(\alpha)$ and hence (2) follows from (5.3). $\square$

6. The Weyl algebras

All the rings we consider below are derived from (localizations of) the Weyl algebras. Hence we discuss these briefly.

We fix some notation which we use throughout. Let $R = k[x_1, \dots, x_r, x_{r+1}^{\pm 1}, \dots, x_{r+s}^{\pm 1}]$ with $r + s = n$. A will be the ring of differential operators of R. That is

$$A = R[\partial_1, \dots, \partial_n]$$

where $\partial_i = \frac{\partial}{\partial x_i}$. We put $\pi_i = x_i \partial_i$, $\mathfrak{t} = k\pi_1 + \cdots + k\pi_n$ and we identify $\mathfrak{t}$, $\mathfrak{t}^*$ with k^n in the obvious way. For $\alpha \in \mathbb{Z}^n \subset \mathfrak{t}^*$ we define

$$u_\alpha = x_1^{(\alpha_1)} \cdots x_r^{(\alpha_r)} x_{r+1}^{\alpha_{r+1}} \cdots x_n^{\alpha_n}$$

where

$$x_i^{(\alpha_i)} = \begin{cases} x_i^{\alpha_i} & \text{if } \alpha_i \geq 0 \\ \partial_i^{-\alpha_i} & \text{if } \alpha_i < 0 \end{cases}$$

Then the $\mathfrak{t}$-weight for the adjoint action of $\mathfrak{t}$ on A is given by α. Furthermore $A = \bigoplus A_0 u_\alpha$ where $A_0 = k[\pi_1, \dots, \pi_n]$. Hence A satisfies the hypotheses (A1)(A2) and therefore we can talk about the $\underset{\alpha}{\Rightarrow}$ and the $\iff$ relation. The result is as follows.

Proposition 6.1. *Let $\alpha, \beta, \gamma \in \mathfrak{t}^*$. Then $\beta \underset{\alpha}{\Rightarrow} \gamma$ iff*

(1) $\alpha \cong \beta \cong \gamma \mod \mathbb{Z}^n$

(2) $\forall i \in \{1, \dots, r\}$ *such that $\alpha_i \in \mathbb{Z}$ one has*

$$\text{If } \alpha_i \geq 0 \text{ and } \beta_i < 0 \text{ then } \gamma_i < 0$$
$$\text{If } \alpha_i < 0 \text{ and } \beta_i \geq 0 \text{ then } \gamma_i \geq 0$$

Proof. According to Proposition 4.1.1 it suffices to look at the cases $A = k[x, \partial]$ and $A = k[x, x^{-1}, \partial]$. In the second case one has $A_m A_n = A_{m+n}$ for all $m, n \in \mathbb{Z}$ and hence by lemma 3.1.9(3) one has $\beta \underset{\alpha}{\Rightarrow} \gamma$ iff $\alpha \cong \beta \cong \gamma \mod \mathbb{Z}^n$.

So we concentrate on the first case. One uses again criterion 3.1.9(3) with

$$\partial x^m = x^{m-1}(x\partial + m)$$
$$x\partial^m = \partial^{m-1}(x\partial - m + 1)$$

for $m \geq 1$. This yields the following basic instances of $\underset{\alpha}{\Rightarrow}$

For $m \geq 1$,

(1) $\alpha + m \underset{\alpha}{\Rightarrow} \alpha + m - 1$ iff $\alpha + m \neq 0$;

(2) $\alpha - m \underset{\alpha}{\Rightarrow} \alpha - m + 1$ iff $\alpha - m + 1 \neq 0$;

and for $m \geq 0$

(3) $\alpha + m \underset{\alpha}{\Rightarrow} \alpha + m + 1$;

(4) $\alpha - m \underset{\alpha}{\Rightarrow} \alpha - m - 1$.

It is easy to see that the above (1)(2)(3)(4) are equivalent with (1)(2) from the statement of the proposition. □

Corollary 6.2. *If $\beta, \gamma \in \mathfrak{t}^*$ then $\beta \iff \gamma$ iff $\beta \cong \gamma \mod \mathbb{Z}^n$ and for all $i \in \{1, \dots, r\}$*

$$\beta_i \in \mathbb{Z} \text{ and } \beta_i \geq 0 \qquad \text{iff} \qquad \gamma_i \in \mathbb{Z} \text{ and } \gamma_i \geq 0$$

Now fix $\theta \in k^n$ and let $\Gamma = \theta + \mathbb{Z}^n$. It is clear that there are only a finite number of equivalence classes for $\Longleftrightarrow$ in Λ. That is condition (A3) is satisfied. Hence we can talk about the orders $H_\Lambda^{(\infty)}$.

Theorem 6.3. *One has*

$$H_\Lambda^{(\infty)} \cong H_1 \mathbin{\hat{\otimes}} H_2 \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} H_n$$

where

$$H_i = \begin{cases} \left(\begin{smallmatrix} k[[\pi_i]] & (\pi_i) \\ k[[\pi_i]] & k[[\pi_i]] \end{smallmatrix}\right) & \text{if } \theta_i \in \mathbb{Z} \text{ and } i \in \{1, \dots, r\} \\ k[[\pi_i]] & \text{otherwise} \end{cases}$$

Proof. Again by Proposition 4.1.1 it suffices to look at the cases $A = k[x, \partial]$ and $A = k[x, x^{-1}, \partial]$. In the second case there is only one equivalence class for $\Longleftrightarrow$ and hence $H_\Lambda^{(\infty)}$ is isomorphic to the completion of A_0 at $\pi - \theta$, which is isomorphic to $k[[\pi]]$.

Assume now that $A = k[x, \partial]$. If $\theta \notin \mathbb{Z}$ then there is again only one equivalence class for $\Longleftrightarrow$. Thus as above $H_\Lambda^{(\infty)} = k[[\pi]]$.

Assume therefore that $\theta \in \mathbb{Z}$. Then it follows cor. 6.2 that there are two equivalence classes : $\langle -1 \rangle$ and $\langle 0 \rangle$. Hence by (3.8)

$$H_\Lambda = \begin{pmatrix} A_0 & A_{-1} \\ A_1 & A_0 \end{pmatrix}$$

with $\psi : D \to H_\Lambda$ given by

$$\pi \mapsto \begin{pmatrix} \pi + 1 & 0 \\ 0 & \pi \end{pmatrix}$$

Conjugation with $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$ transforms H_Λ in

$$\begin{pmatrix} A_0 & (\pi) \\ A_0 & A_0 \end{pmatrix} \tag{6.1}$$

and ψ becomes the diagonal map $a \mapsto \operatorname{diag}(a, a)$. Completing (6.1) at (π) yields the desired result. $\square$

Remark 6.4. It helps to observe that $k[[\pi]]$ is the completed path algebra of the one-loop quiver

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$$

and $\left(\begin{smallmatrix} k[[\pi]] & (\pi) \\ k[[\pi]] & k[[\pi]] \end{smallmatrix}\right)$ is the completed path algebra of

$$\begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \end{tikzcd}$$

Hence the general case is obtained by taking a product of such quivers (in an appropriate sense). In this way one obtains a quiver whose finite dimensional representations describe the finitely generated objects in $\mathcal{O}_\Lambda^{(\infty)}$.

7. Rings of differential operators for torus invariants

Let $A, \mathfrak{t}, R, \dots$ be as in §6. Assume that G is an algebraic torus acting diagonally on $kx_1 + \cdots + kx_r + kx_{r+1} + \cdots + kx_{r+s}$ with weights $\eta_1, \dots, \eta_n \in X(G)$, where $n = r + s$. We may and we will assume that the generic stabilizer for the G-action is trivial. That is, we assume that the weights $\eta_1, \dots, \eta_n$ span $X(G)$.

Restriction of differential operators provides us with a natural map

$$D(R)^G/\mathfrak{g}D(R)^G \to D(R^G) \tag{7.1}$$

where $D(-)$ denotes the ring of differential operators [18][22]. This map is usually an isomorphism, but the exact conditions under which this happens are somewhat technical. See loc. sit.

Even if (7.1) is not surjective then one may find a new ring of Laurent polynomials R' with an action of a new torus G' such that $D(R^G) = D(R'^{G'})$ and such that (7.1) is an isomorphism with (G, R) replaced by (G', R').

The map (7.1) may be generalized to covariants. Let $\chi \in X(G)$. Then the R^G-module of co (or semi) invariants associated to χ is

$$R^G_\chi = \{r \in R \mid \forall g \in G : g \cdot r = \chi(g)r\}$$

Let $\mathfrak{g} = \operatorname{Lie} G$. The fact that the generic stabilizer for the G-action is trivial implies in particular that $\mathfrak{g} \subset \mathfrak{t}$. We may canonically embed $X(G) \subset \mathfrak{g}^*$ and hence χ may be considered as an element of $\mathfrak{g}^*$. Then there is again a natural map

$$D(R)^G/(\mathfrak{g} - \chi(\mathfrak{g}))D(R)^G \to D_{R^G}(R^G_\chi) \tag{7.2}$$

which is usually an isomorphism. We recall that $\mathfrak{g} - \chi(\mathfrak{g})$ was defined in (4.4).

Below we study the left hand side of (7.2), but we will not restrict χ to being an element of $X(G)$. That is, χ will be an arbitrary element of $\mathfrak{g}^*$. Working in this greater generality is essentially for free.

If we put, as in §6, $A = D(R)$ then, using the notation of §4.4, we have

$$B^\chi = A^G/(\mathfrak{g} - \chi(\mathfrak{g}))A^G$$

The rings B^χ also turn up in the study of rings of differential global operators on toric varieties, see [19].

We will call $\mathfrak{h} \subset \mathfrak{t}$ algebraic if it is the Lie algebra of some algebraic torus, acting diagonally on $\sum_i kx_i$. This equivalent with

$$\mathfrak{h} = \bigcap_i \ker \lambda_i$$

for some $(\lambda_i)_i \in \mathbb{Q}^n \subset \mathfrak{t}^*$.

7.1. A few results on Zariski closures.

If one wants to apply the results from §3 to rings of differential operators, the main difficulty consists of describing the regions $\overline{\langle \alpha \rangle}$, and more specifically checking the conditions for Theorem 3.2.4. In this section we provide some results which are related to this. We use some standard results and notations from convex geometry for which we refer the reader to [21].

Lemma 7.1.1. *Assume that E is a finite dimensional F-vector space, F a subfield of $\mathbb{R}$, and let $\lambda_1, \dots, \lambda_m \in E^*$. Then there exists a disjoint decomposition*

$$\{1, \dots, m\} = I \sqcup J \tag{7.3}$$

such that there exist $\epsilon \in E$ and $z \in F^m$ with the properties $\sum_{i=1}^m z_i\lambda_i = 0$,

$$\langle \lambda_i, \epsilon\rangle = \begin{cases} > 0 & \text{if } i \in I \\ = 0 & \text{if } i \in J \end{cases}$$

$$z_i = \begin{cases} = 0 & \text{if } i \in I \\ > 0 & \text{if } i \in J \end{cases}$$

Furthermore, a decomposition (7.3)*, with the property that ϵ, z exist, is unique.*

Proof. Let T be the positive span of $(\lambda_i)_{i=1,\dots,m}$, $H = T \cap (-T)$ the maximal linear subspace of T, $C = T^\vee$. Then $H = C^\perp$. Let $J = \{i \mid \lambda_i \in H\}$, $I = \{i \mid \lambda_i \in C^\vee \setminus C^\perp\}$ and choose $\epsilon \in \operatorname{relint}(C)$. Then [21, Lemma A.4] gives $\langle \lambda_i, \epsilon\rangle > 0$ iff $i \in I$.

Also if $j \in J$ then $\mathbb{R}_+(-\lambda_j) \subset T$ and so we can find $z_i^{(j)} \in F$ such that $\sum_i z_i^{(j)}\lambda_i = 0$, $z_i^{(j)} \geq 0$ for all i and $z_j^{(j)} > 0$. set $z_i = \sum_j z_i^{(j)}$. Then $\sum z_i\lambda_i = 0$, $z_i \geq 0$ for all i and $z_i > 0$ for $i \in J$. The existence of ϵ forces $z_i = 0$ for $i \in I$.

The uniqueness of the decomposition is proved similarly. □

Proposition 7.1.2. *Let E be a $\mathbb{Q}$-vector space, L a full $\mathbb{Z}$-lattice in E, $\lambda_1, \dots, \lambda_m \in E^*$, $c_1, \dots, c_m \in \mathbb{Q}$, and define*

$$C = \{x \in E \mid \forall i = 1, \dots, m : \langle \lambda_i, x\rangle \leq c_i\}$$

Let $\{1, \dots, m\} = I \sqcup J$ be a decomposition as in (7.3)*. Put*

$$E' = \bigcap_{j\in J} \ker \lambda_j$$

$$C' = \{x \in E \mid \forall i \in J : \langle \lambda_i, x\rangle \leq c_i\}$$

Then

(1) $\overline{C \cap L} = C' \cap (L + E')$

(2) *$C' \cap (L + E')$ is a finite union of translates of E'.*

Proof. Throughout let ϵ, z be as in lemma 7.1.1. We first prove (2). We define a map

$$\iota : C' \cap (L + E') \to \mathbb{Q}^{|J|} : x \mapsto (\langle \lambda_j, x\rangle)_{j\in J}$$

By definition if $x \in C'$ then $\langle \lambda_j, x\rangle \leq c_j$ for all $j \in J$. On the other hand

$$\lambda_j = -\frac{1}{z_j} \sum_{\substack{k\neq j \\ k\in J}} z_k\lambda_k$$

and hence

$$\begin{aligned} \langle \lambda_j, x\rangle &= -\frac{1}{z_j} \sum_{\substack{k\neq j \\ k\in J}} z_k\langle \lambda_k, x\rangle \\ &\geq -\frac{1}{z_j} \sum_{\substack{k\neq j \\ k\in J}} z_k c_k \end{aligned}$$

Thus the image of ι is bounded.

In addition we have $\operatorname{im} \iota = \iota(C' \cap L)$, and hence the image of ι is discrete (E is a $\mathbb{Q}$-vector space). Together this shows that $\operatorname{im} \iota$ is finite. Since the fibers of ι consist of translates of E', (2) is proved.

Now we prove (1). Note that $C \cap L \subset C' \cap (L + E')$. Using (2) it suffices now to prove two statements :

(1a) $\iota(C \cap L) = \operatorname{im} \iota$.

Because L is dense in E we may replace ϵ by some positive multiple such that $\epsilon \in L$. Assume that $l + e \in C' \cap (L + E')$ with $l \in L$, $e \in E'$. Then $\iota(l - M\epsilon) = \iota(l) = \iota(l + e)$, for $M \in \mathbb{N}$. If we choose $M \gg 0$ then $l - M\epsilon \in C \cap L$.

(1b) If $\zeta \in \operatorname{im} \iota$ then $\iota^{-1}(\zeta) \cap (C \cap L)$ is Zariski dense in $\iota^{-1}(\zeta)$.

To prove this let $(\lambda_i')_{i=1,\ldots,m}$ be the restrictions of $(\lambda_i)_{i=1,\ldots,m}$ to E'. Let $x \in \iota^{-1}(\zeta) \cap (C \cap L)$. Then $\iota^{-1}(\zeta) \cap (C \cap L) = x + U$ where

$$U = \{y \in E' \cap L \mid \langle \lambda_i', x + y\rangle \le c_i \text{ for all } i \in I\}$$

Note now that by definition ϵ is in E' and satisfies $\langle \lambda_i', \epsilon\rangle > 0$ for all $i \in I$. Furthermore $L \cap E'$ is a dense sublattice of E' since the ground field is $\mathbb{Q}$. Then it follows from [28, lemma 3.4] that U is Zariski dense in E'. Since $\iota^{-1}(\zeta) = x + E'$ this proves (1b). □

Remark 7.1.3.
- Proposition 7.1.2 is false if $\mathbb{Q}$ is replaced by $\mathbb{R}$.
- Since for $\lambda \in E^*$, $\langle \lambda, - \rangle$ takes on discrete values, we may replace some of the $\le$-signs in the definition of C by $<$-signs, provided we do the same with the corresponding signs in the definition of C'.

Corollary 7.1.4. *Let $x \in L$, Then*

$$\overline{(x + C \cap L) \cap (C \cap L)} = \overline{x + C \cap L} \cap \overline{C \cap L} \tag{7.4}$$

scheme theoretically.

Proof. It suffices to prove (7.4) set theoretically since we are talking about finite unions of translates of linear spaces. By Proposition 7.1.2 the right-hand side of (7.4) is equal to

$$(x + C') \cap C' \cap (L + E')$$

Let $a_i = \min(c_i, c_i - \langle \lambda, x\rangle)$ for $i = 1, \ldots, m$. Then

$$(x + C) \cap C = \{y \in E \mid \forall i = 1, \ldots, m : \langle \lambda_i, y\rangle \le a_i\}$$

Let $((x + C) \cap C)'$ be derived from $(x + C) \cap C$ in the same way as C' is derived from C. That is

$$((x + C) \cap C)' = \{y \in E \mid \forall i \in J : \langle \lambda_i, y\rangle \le a_i\}$$

It is then easy to see that the left-hand side of (7.4) is equal to

$$((x + C) \cap C)' \cap (L + E')$$

Hence we have to show

$$((x + C) \cap C)' = (x + C') \cap C'$$

but this is clear. □

The following result will be used later

Lemma 7.1.5. *Let X be a set of the form $C \cap L$ (as in 7.1.2) and assume that X is Zariski dense in E. Then given $\beta, \gamma \in L$ there exist $\alpha \in X$ such that $\alpha + \gamma \in X$, $\alpha + \beta + \gamma \in X$.*

Proof. According to [28, lemma 3.4] there exists ϵ such that for all $i \in \{1, \dots, m\}$: $\langle \lambda_i, \epsilon \rangle > 0$. By replacing ϵ with a suitable multiple, we may assume that $\epsilon \in L$. It then suffices to take $\alpha = -M\epsilon$ for $M \gg 0$. □

7.2. **Computation of $\overline{\langle \alpha \rangle}_{B^\chi}$.** In this section we let the notation be as in the beginning of §7. We will use the results of §7.1 to give a classification of the $\overline{\langle \alpha \rangle}_{B^\chi}$. Although not strictly indispensable, this result will be useful in the classification of the primitive ideals of B^χ which we give in §7.3. The reader is advised to read that section first.

We recall that $\eta_1, \dots, \eta_n \in \mathfrak{g}^*$ are the weights for the action of $\mathfrak{g}$ on $kx_1 + \cdots + kx_n$. We have $\mathfrak{g} \subset \mathfrak{t}$ and we identify $\mathfrak{t}, \mathfrak{t}^*$ with k^n as in §6. Then

$$V(\mathfrak{g} - \chi(\mathfrak{g})) = \{(\alpha_i)_{i=1,\dots,n} \in \mathfrak{t}^* \mid \sum_{i=1}^n \alpha_i \eta_i = \chi\}$$

Definition 7.2.1. Let (ψ, θ) be a pair satisfying

(1) $\psi \in \mathfrak{g} \cap \mathbb{Q}^n$
(2) $\langle \psi, \eta_i \rangle = 0$ for $i \notin \{1, \dots, r\}$
(3) $\theta \in \left(\sum_{\langle \psi, \eta_i \rangle = 0} k\eta_i\right) / \left(\sum_{\langle \psi, \eta_i \rangle = 0} \mathbb{Z}\eta_i\right)$

Then we say that (ψ, θ) is attached to χ if there exist $\beta \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ with the properties

(4) $\sum_{\langle \psi, \eta_i \rangle = 0} \beta_i \eta_i \cong \theta \mod \sum_{\langle \psi, \eta_i \rangle = 0} \mathbb{Z}\eta_i$
(5) For all $i \in \{1, \dots, r\}$

$$\langle \psi, \eta_i \rangle < 0 \Rightarrow \beta_i \in \mathbb{Z}, \beta_i \geq 0$$
$$\langle \psi, \eta_i \rangle > 0 \Rightarrow \beta_i \in \mathbb{Z}, \beta_i < 0$$
$$\langle \psi, \eta_i \rangle = 0 \Rightarrow \beta_i \notin \mathbb{Z}$$

Remark 7.2.2. (1) Property (5) above makes sense since $\psi \in \mathfrak{g} \cap \mathbb{Q}^n$ and $\eta_i \in X(G)$ which is in the image of $\mathbb{Z}^n \subset \mathfrak{t}^*$ in $\mathfrak{g}^*$. Hence $\langle \psi, \eta_i \rangle \in \mathbb{Q}$.
(2) Properties (1)(2)(3) of (ψ, θ) are independent of χ. For a given pair (ψ, θ) to be attached to at least one χ it is necessary and sufficient that θ is in the image of

$$\sum_{\substack{i \in \{1,\dots,r\} \\ \langle \psi, \eta_i \rangle = 0}} (k - \mathbb{Z})\eta_i + \sum_{i \notin \{1,\dots,r\}} k\eta_i$$

To a pair (ψ, θ) satisfying (1)(2)(3) we associate a set

$$S_{\psi,\theta} = \left\{ \begin{array}{c} (\gamma_i)_i \in V(\mathfrak{g} - \chi(\mathfrak{g})) \mid \sum_{\langle \psi, \eta_i \rangle = 0} \gamma_i \eta_i \cong \theta \mod \sum_{\langle \psi, \eta_i \rangle = 0} \mathbb{Z}\eta_i \\ \text{and } \forall i \in \{1, \dots, n\} : \begin{array}{l} \langle \psi, \eta_i \rangle < 0 \Rightarrow \gamma_i \in \mathbb{Z}, \gamma_i \geq 0 \\ \langle \psi, \eta_i \rangle > 0 \Rightarrow \gamma_i \in \mathbb{Z}, \gamma_i < 0 \end{array} \end{array} \right\} \tag{7.5}$$

Lemma 7.2.3. *$S_{\psi,\theta}$ is a finite union of translates of the linear space*

$$\{(u_i)_{i=1,\dots,n} \in V(\mathfrak{g}) \mid \forall i \in \{1, \dots, n\} : \langle \psi, \eta_i \rangle \neq 0 \Rightarrow u_i = 0\} \tag{7.6}$$

Proof. This result can be obtained from the proof of Proposition 7.2.4. However for clarity we give an independent proof.

Define the map

$$\iota : S_{\psi,\theta} \to \mathbb{Z}^v : (\gamma_i) \mapsto (\gamma_i)_{\langle \psi, \eta_i \rangle \neq 0}$$

where $v = |\{i \mid \langle\psi,\eta_i\rangle \neq 0\}|$. The fibers of ι clearly consist of translates of (7.6).

Let $(\gamma_i)_i \in S_{\psi,\theta}$. Then

$$\sum_{\langle\psi,\eta_i\rangle\neq 0} \gamma_i\eta_i = \chi - \theta \bmod \sum_{\langle\psi,\eta_i\rangle=0} \mathbb{Z}\eta_i$$

Applying $\langle\psi,-\rangle$ and using (3) in definition 7.2.1 yields

$$\sum_{\langle\psi,\eta_i\rangle\neq 0} \gamma_i\langle\psi,\eta_i\rangle = \langle\psi,\chi\rangle$$

Now if $\langle\psi,\eta_i\rangle \neq 0$ then the definition of $S_{\psi,\theta}$ implies that $\gamma_i \in \mathbb{Z}$ and $\gamma_i\langle\psi,\eta_i\rangle \leq 0$. Thus there are only finitely many possibilities for $(\gamma_i)_{\langle\psi,\eta_i\rangle\neq 0}$. Whence the image of ι is finite. □

The following proposition is the main result of this section

Proposition 7.2.4. *Every $\overline{\langle\alpha\rangle}_{B\chi}$ is of the form $S_{\psi,\theta}$ where (ψ,θ) is attached to χ. Conversely, if (ψ,θ) is attached to χ then $S_{\psi,\theta} = \overline{\langle\beta\rangle}_{B\chi}$ where β is as in definition 7.2.1(4)(5).*

We will prove this result below. The following proposition tells us when $S_{\psi,\theta} \subset S_{\psi',\theta'}$ and when $S_{\psi,\theta} = S_{\psi',\theta'}$

Proposition 7.2.5. *Assume that $(\psi,\theta),(\psi',\theta')$ are attached to χ. Then*

(1) $S_{\psi,\theta} \subset S_{\psi',\theta'}$ *iff*

$$\begin{aligned}\{i \mid \langle\psi',\eta_i\rangle < 0\} &\subset \{i \mid \langle\psi,\eta_i\rangle < 0\}\\ \{i \mid \langle\psi',\eta_i\rangle > 0\} &\subset \{i \mid \langle\psi,\eta_i\rangle > 0\}\end{aligned} \tag{7.7}$$

$$\theta' \cong \theta \quad \text{mod} \sum_{\langle\psi',\eta_i\rangle=0} \mathbb{Z}\eta_i \tag{7.8}$$

(2) $S_{\psi,\theta} = S_{\psi',\theta'}$ *iff*

$$\begin{aligned}\{i \mid \langle\psi',\eta_i\rangle < 0\} &= \{i \mid \langle\psi,\eta_i\rangle < 0\}\\ \{i \mid \langle\psi',\eta_i\rangle > 0\} &= \{i \mid \langle\psi,\eta_i\rangle > 0\}\\ \theta \cong \theta' &\quad \text{mod} \sum_{\langle\psi,\eta_i\rangle=0} \mathbb{Z}\eta_i\end{aligned}$$

Proof. (2) follows from (1), so we concentrate on (1). If (7.7)(7.8) hold then clearly $S_{\psi,\theta} \subset S_{\psi',\theta'}$, so we prove the converse. Assume $S_{\psi,\theta} \subset S_{\psi',\theta'}$. Since (ψ,θ) is attached to χ there exist β satisfying 7.2.1(4)(5). We deduce

$$\theta \cong \sum_{\langle\psi,\eta_i\rangle=0} \beta_i\eta_i \quad \text{mod} \sum_{\langle\psi,\eta_i\rangle=0} \mathbb{Z}\eta_i \tag{7.9}$$

$$\begin{aligned}\{i \mid \langle\psi,\eta_i\rangle < 0\} &= \{i \in \{1,\ldots,r\} \mid \beta_i \in \mathbb{Z}, \beta_i \geq 0\}\\ \{i \mid \langle\psi,\eta_i\rangle > 0\} &= \{i \in \{1,\ldots,r\} \mid \beta_i \in \mathbb{Z}, \beta_i < 0\}\end{aligned} \tag{7.10}$$

Now $\beta \in S_{\psi',\theta'}$ and hence by (7.5)

$$\theta' \cong \sum_{\langle\psi',\eta_i\rangle=0} \beta_i\eta_i \quad \text{mod} \sum_{\langle\psi',\eta_i\rangle=0} \mathbb{Z}\eta_i \tag{7.11}$$

$$\begin{aligned}\{i \mid \langle\psi',\eta_i\rangle < 0\} &\subset \{i \in \{1,\ldots,r\} \mid \beta_i \in \mathbb{Z}, \beta_i \geq 0\}\\ \{i \mid \langle\psi',\eta_i\rangle > 0\} &\subset \{i \in \{1,\ldots,r\} \mid \beta_i \in \mathbb{Z}, \beta_i < 0\}\end{aligned} \tag{7.12}$$

From (7.10)(7.12) we deduce

$$\{i \mid \langle\psi',\eta_i\rangle < 0\} \subset \{i \mid \langle\psi,\eta_i\rangle < 0\}$$
$$\{i \mid \langle\psi',\eta_i\rangle > 0\} \subset \{i \mid \langle\psi,\eta_i\rangle > 0\}$$

Furthermore (7.10) implies that

$$\sum_{\langle\psi',\eta_i\rangle=0} \beta_i\eta_i \cong \sum_{\langle\psi,\eta_i\rangle=0} \beta_i\eta_i \mod \sum_{\langle\psi',\eta_i\rangle=0} \mathbb{Z}\eta_i$$

This yields (7.8). □

Corollary 7.2.6. *There are only a finite number of different* $\overline{\langle\alpha\rangle}_{B^\chi}$.

Proof. The proof is similar to that of lemma 7.2.3. By Propositions 7.2.4 and 7.2.5 it suffices to show that for every ψ satisfying (1)(2) of definition 7.2.1, there are only finitely many θ such that (ψ,θ) is attached to χ. Suppose that we are given θ and $\beta \in V(\mathfrak{g}-\chi(\mathfrak{g}))$ such that (3)(4)(5) of definition 7.2.1 hold. Then

$$\sum_{\langle\psi,\eta_i\rangle\neq 0} \beta_i\eta_i = \chi - \theta \mod \sum_{\langle\psi,\eta_i\rangle=0} \mathbb{Z}\eta_i$$

Applying $\langle\psi,\ \rangle$ and using (2) and (3) we obtain

$$\sum_{\langle\psi,\eta_i\rangle\neq 0} \beta_i\langle\psi,\eta_i\rangle = \langle\psi,\chi\rangle \tag{7.13}$$

Now if $\langle\psi,\eta_i\rangle \neq 0$ then (5) implies that $\beta_i \in \mathbb{Z}$ and $\beta_i\langle\psi,\eta_i\rangle \leq 0$. Thus there are only finitely many possibilities for $(\beta_i)_{\langle\psi,\eta_i\rangle\neq 0}$ satisfying (7.13), and hence only finitely many possibilities for θ. □

Example 7.2.7. The above proof gives a method for calculating all θ such that (ψ,θ) is attached to χ. We give an example which shows that ψ does not determine θ. We consider the action of a 2-dimensional torus G on $kx_1+\cdots+kx_4$. We identify $X(G)$ with $\mathbb{Z}^2$. Suppose the weights $\eta_1,\ldots,\eta_4$ are given by $(0,2)$, $(0,-2)$, $(1,0)$, $(1,1)$. Let $\chi=(1,2)$ and $\psi=(-1,0)$. Then (7.13) becomes

$$-\beta_3-\beta_4=-1$$

with β_3,β_4 nonnegative integers, with solutions $(\beta_3,\beta_4)=(1,0)$ or $(0,1)$. The corresponding values of θ are $\chi-\eta_3=(0,2)$ and $\chi-\eta_4=(0,1)$ which lie in distinct cosets of $\sum_{\langle\psi,\eta_i\rangle=0}\mathbb{Z}\eta_i = \mathbb{Z}(0,2)$. Finally we can choose β_1,β_2 so that $\beta=(1/2,-1/2,1,0)$ in the first case and $\beta=(1/4,-1/4,0,1)$ in the second case.

Proof of Proposition 7.2.4. First we fix $\alpha\in V(\mathfrak{g}-\chi(\mathfrak{g}))$. For simplicity we write $\langle\alpha\rangle$ for $\langle\alpha\rangle_{B^\chi}$. We also put $T=\{1,\ldots,r\}\cap\{i\mid\alpha_i\in\mathbb{Z}\}$. Then by Corollary 6.2

$$\langle\alpha\rangle = \left\{\beta\in V(\mathfrak{g}-\chi(\mathfrak{g})) \mid \beta\cong\alpha \mod \mathbb{Z}^n \quad\text{and}\quad \forall i\in T: \begin{matrix}\alpha_i\geq 0\Rightarrow\beta_i\geq 0\\ \alpha_i<0\Rightarrow\beta_i<0\end{matrix}\right\}$$

We will write $\langle\alpha\rangle-\alpha$ in the form $C\cap L$ as in §7.1. More precisely we put $E=V(\mathfrak{g})\cap\mathbb{Q}^n$, $L=\operatorname{Supp}B^\chi=\operatorname{Supp}A\cap E$ and for $i\in T$, λ_i is defined by

$$\alpha_i\geq 0\Rightarrow\lambda_i(u)=-u_i$$
$$\alpha_i<0\Rightarrow\lambda_i(u)=u_i$$

If we put

$$C = \left\{ u \in V(\mathfrak{g}) \mid \forall i \in T : \begin{array}{l} \alpha_i \geq 0 \Rightarrow \lambda_i(u) \leq \alpha_i \\ \alpha_i < 0 \Rightarrow \lambda_i(u) \leq -\alpha_i - 1 \end{array} \right\}$$

then we indeed have

$$\langle \alpha \rangle - \alpha = L \cap C$$

Note that the fact that $\mathfrak{g} = \operatorname{Lie} G$ implies that L is a full sublattice in E.

Now using lemma 7.1.1, let $T = I \sqcup J$ be a disjoint decomposition such that there exist $\epsilon \in E$ and $z \in \mathbb{Q}^{|T|}$ such that $\sum z_i \lambda_i = 0$ and

$$\langle \lambda_i, \epsilon \rangle > 0 \text{ and } z_i = 0 \text{ if } i \in I$$
$$\langle \lambda_i, \epsilon \rangle = 0 \text{ and } z_i > 0 \text{ if } i \in J$$

Define $(y_i)_{i=1,\ldots,n} \in \mathbb{Q}^n$ by

$$y_i = \begin{cases} -z_i & \text{if } i \in T,\ \alpha_i \geq 0 \\ z_i & \text{if } i \in T,\ \alpha_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

If $(\omega_i)_i \in \mathbb{Q}^n$ satisfies $\sum \omega_i \eta_i = 0$ then $(\omega_i)_i \in E$. Evaluating $\sum z_i \lambda_i$ on $(\omega_i)_i$ yields $\sum \omega_i y_i = 0$. Since this holds for all such $(\omega_i)_i$ and $\eta_i \in \mathfrak{g} \cap \mathbb{Q}^n$ this implies that there must exist $\psi \in \mathfrak{g} \cap \mathbb{Q}^n$ such that $y_i = \langle \psi, \eta_i \rangle$.

Now we use lemma 7.1.2 to compute the Zariski closure of $C \cap L$ in E. Note that for $i \in \{1, \ldots, n\}$ we have $y_i = \langle \psi, \eta_i \rangle \neq 0$ if and only if $i \in J$. Thus

$$E' = \bigcap_{j \in J} \ker \lambda_j = \{(u_i)_{i=1,\ldots,n} \in V(\mathfrak{g}) \cap \mathbb{Q}^n \mid \forall i \in \{1, \ldots, n\} : \langle \psi, \eta_i \rangle \neq 0 \Rightarrow u_i = 0\}$$

Furthermore, if $\langle \psi, \eta_i \rangle < 0$ then $\alpha_i \geq 0$ and the condition $\lambda_i(u) \leq \alpha_i$ in the definition of C is equivalent to $u_i \geq -\alpha_i$. Similarly if $\langle \psi, \eta_i \rangle > 0$ then $\alpha_i < 0$ and the condition $\lambda_i(u) \leq -\alpha_i - 1$ becomes $u_i < -\alpha_i$. Hence

$$C' = \left\{ (u_i)_{i=1,\ldots,n} \in V(\mathfrak{g}) \cap \mathbb{Q}^n \mid \forall i \in \{1, \ldots, n\} : \begin{array}{l} \langle \psi, \eta_i \rangle < 0 \Rightarrow u_i \geq -\alpha_i \\ \langle \psi, \eta_i \rangle > 0 \Rightarrow u_i < -\alpha_i \end{array} \right\}$$

Now by Proposition 7.1.2

$$\overline{C \cap L} = (E' + L) \cap C'$$

Since

$$E' + L = \left\{ \begin{array}{l} (u_i)_i \in V(\mathfrak{g}) \cap \mathbb{Q}^n \mid \exists (v_i)_i \in V(\mathfrak{g}) \cap \mathbb{Z}^n \\ \qquad \text{such that } \forall i \in \{1, \ldots, n\} : \langle \psi, \eta_i \rangle \neq 0 \Rightarrow u_i = v_i \end{array} \right\}$$

we find

$$\overline{C \cap L} = \left\{ \begin{array}{l} (u_i)_i \in V(\mathfrak{g}) \cap \mathbb{Q}^n \mid \exists (v_i)_i \in V(\mathfrak{g}) \cap \mathbb{Z}^n \\ \qquad \text{such that } \forall i \in \{1, \ldots, n\} : \begin{array}{l} \langle \psi, \eta_i \rangle < 0 \Rightarrow u_i = v_i \geq -\alpha_i \\ \langle \psi, \eta_i \rangle > 0 \Rightarrow u_i = v_i < -\alpha_i \end{array} \end{array} \right\}$$

This is the $\mathbb{Q}$-Zariski closure of $\langle \alpha \rangle - \alpha$. To find $\overline{\langle \alpha \rangle}$ we have to take the k-Zariski closure and add α. We find

$$\overline{\langle \alpha \rangle} = \left\{ \begin{array}{l} (\gamma_i)_i \in V(\mathfrak{g} - \chi(\mathfrak{g})) \mid \exists (\delta_i)_i \in V(\mathfrak{g} - \chi(\mathfrak{g})) : \delta \cong \alpha \mod \mathbb{Z}^n \\ \qquad \text{and } \forall i \in \{1, \ldots, n\} : \begin{array}{l} \langle \psi, \eta_i \rangle < 0 \Rightarrow \gamma_i = \delta_i \geq 0 \\ \langle \psi, \eta_i \rangle > 0 \Rightarrow \gamma_i = \delta_i < 0 \end{array} \end{array} \right\} \tag{7.14}$$

It will be useful to rewrite (7.14) a bit. The existence of $\delta \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ such that $\delta \cong \alpha \mod \mathbb{Z}^n$ and $\langle \psi, \eta_i \rangle \neq 0 \Rightarrow \gamma_i = \delta_i$ is equivalent to

$$\sum_{\langle \psi, \eta_i \rangle = 0} \gamma_i \eta_i \in \sum_{\langle \psi, \eta_i \rangle = 0} \alpha_i \eta_i + \sum_{\langle \psi, \eta_i \rangle = 0} \mathbb{Z} \eta_i$$

Hence if θ is the image of $\sum_{\langle \psi, \eta_i \rangle = 0} \alpha_i \eta_i$ in

$$\left(\sum_{\langle \psi, \eta_i \rangle = 0} k \eta_i \right) / \left(\sum_{\langle \psi, \eta_i \rangle = 0} \mathbb{Z} \eta_i \right)$$

then we obtain from (7.14) :

$$\overline{\langle \alpha \rangle} = S_{\psi, \theta}$$

Now by construction (ψ, θ) satisfies 7.2.1(1)(2)(3). Furthermore for $\mu \in \mathbb{Q}$ one has

$$\sum_{\langle \psi, \eta_i \rangle = 0} \alpha_i \eta_i = \sum_{\langle \psi, \eta_i \rangle = 0} (\alpha_i + \mu \epsilon_i) \eta_i$$

Thus $\beta = \alpha + \mu\epsilon$ satisfies 7.2.1(4) and since for all $i \in T$ we have $\langle \psi, \eta_i \rangle \neq 0$ if and only if $\epsilon_i = 0$ we see that β satisfies 7.2.1(5) for μ small enough. This shows that (ψ, θ) is attached to χ.

Now we indicate how one proves the converse. Assume that (ψ, θ) is attached to χ. Then we claim that $S_{\psi,\theta} = \overline{\langle \beta \rangle}$ where β is as in definition 7.2.1. This follows by retracing the computations in the first part of the proof. It turns out that one has to take $I = \emptyset$, $J = T$, $\epsilon = 0$, $z_i = |\langle \psi, \eta_i \rangle|$. □

7.3. **Primitive ideals.** In this section we will verify the hypotheses for Theorem 3.2.4 and Propositions 3.2.2 and 3.4.1 for B^χ as introduced in the beginning of §7. For simplicity we restate a combined version of these results below.

Theorem 7.3.1. (1) *B^χ is a domain.*
(2) *B^χ is primitive.*
(3) *Every prime ideal in B^χ is of the form $J(\alpha)$ with $\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))$. In particular every prime ideal is primitive.*
(4) *There is a one-one correspondence between the regions $\overline{\langle \alpha \rangle}_{B^\chi} \subset V(\mathfrak{g} - \chi(\mathfrak{g}))$ and the primitive ideals in B^χ. The correspondence is given by associating $J(\alpha)$ to $\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))$.*
(5) *B^χ has only a finite number of primitive ideals;*
(6) *If J is a primitive ideal in B^χ then $J_\alpha = B^\chi_\alpha J_0 + J_0 B^\chi_\alpha$. In particular, J is generated in degree zero.*

Proof. Let $\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))$. We write $\langle \alpha \rangle$ for $\langle \alpha \rangle_{B^\chi}$. Then

$$\langle \alpha \rangle = \left\{ \begin{array}{ll} \beta \in V(\mathfrak{g} - \chi(\mathfrak{g})) \mid \beta \cong \alpha \mod \mathbb{Z}^n & \\ \text{and } \forall i \in \{1, \ldots, r\}, \alpha_i \in \mathbb{Z} : & \begin{array}{l} \alpha_i \geq 0 \Rightarrow \beta_i \geq 0 \\ \alpha_i < 0 \Rightarrow \beta_i < 0 \end{array} \end{array} \right\} \tag{7.15}$$

In the course of the proof of Proposition 7.2.4 we have shown that $\langle \alpha \rangle - \alpha$ may be written in the form $C \cap L$, where C, L are as in Proposition 7.1.2. Hence for $\beta \in \operatorname{Supp} B^\chi$ we obtain by cor. 7.1.4

$$\overline{(\langle \alpha \rangle - \alpha + \beta) \cap (\langle \alpha \rangle - \alpha)} = \overline{\langle \alpha \rangle - \alpha + \beta} \cap \overline{\langle \alpha \rangle - \alpha}$$

Translating this by α we obtain

$$\overline{(\langle\alpha\rangle+\beta)\cap\langle\alpha\rangle}=\overline{\langle\alpha\rangle+\beta}\cap\overline{\langle\alpha\rangle}$$

which was the hypothesis for Proposition 3.2.2. So this proves that every $J(\alpha)$ is generated in degree zero and is determined by $\overline{\langle\alpha\rangle}$.

Fix $\mu\in V(\mathfrak{g}-\chi(\mathfrak{g}))$ and put $\Lambda=\mu+\operatorname{Supp}B^\chi$. From the fact that $\mathfrak{g}=\operatorname{Lie}G$, one obtains that $\operatorname{Supp}B^\chi$ is a full lattice in $V(\mathfrak{g})$. Hence $\overline{\Lambda}=\mu+\operatorname{Supp}B^\chi$. By (7.15) it follows that there are only a finite number of equivalence classes for $\Longleftrightarrow$ in Λ. Hence at least one of those must be dense. This proves (2).

Let $\langle\alpha\rangle$ be such a Zariski dense equivalence class in Λ. Since $\langle\alpha\rangle$ is (up to translation) of the form $C\cap L$ it follows from lemma 7.1.5 that for all $\beta,\gamma\in\operatorname{Supp}B^\chi$ there exist $\delta\in\langle\alpha\rangle$ such that $\delta+\gamma\in\langle\alpha\rangle$, $\delta+\beta+\gamma\in\langle\alpha\rangle$. Then (1) follows from Proposition 3.4.1. The only remaining non-trivial hypothesis we have to verify is 3.2.4(3). But this is precisely corollary 7.2.6. □

Remark 7.3.2.
1. Note that the $\overline{\langle\alpha\rangle}_{B^\chi}$ were described in §7.2. Hence it follows from 7.3.1(4) that there is also a one-one correspondence between the equivalence classes of (ψ,θ)'s attached to χ and primitive ideals in B^χ (where the equivalence relation is deduced from Proposition 7.2.5)
2. In this context it is useful to observe that in a pair (ψ,θ) one may choose ψ in $\mathfrak{g}\cap\mathbb{Z}^n=Y(G)$. Hence primitive ideals in B^χ are to a certain extent determined by one-parameter subgroups of G.

7.4. **Primitive quotients.** It is possible to describe the primitive quotients of B^χ. In this section we will write $B^\chi_{\mathfrak{g}}$ for B^χ.

Proposition 7.4.1. *Assume that J is a primitive ideal in $B^\chi_{\mathfrak{g}}$. Then there exist an algebraic $\mathfrak{g}\subset\mathfrak{h}\subset\mathfrak{t}$ and $\chi_1,\dots,\chi_p\in\mathfrak{h}^*$, $\forall i=1,\dots,p:\chi_i|_{\mathfrak{g}}=\chi$ such that*

$$B^\chi_{\mathfrak{g}}/J=\begin{pmatrix} B^{\chi_1}_{\mathfrak{h}} & B^{\chi_1,\chi_2}_{\mathfrak{h}} & & \\ B^{\chi_2,\chi_1}_{\mathfrak{h}} & B^{\chi_2}_{\mathfrak{h}} & & \\ & & \ddots & \\ & & & B^{\chi_p}_{\mathfrak{h}} \end{pmatrix}$$

Proof. By Theorem 7.3.1(3), $J=J(\alpha)$ and hence $J_0=I\big(\overline{\langle\alpha\rangle}\big)$. By Proposition 7.1.2 and lemma 7.2.3

$$\overline{\langle\alpha\rangle}=V(\mathfrak{h}-\chi_1(\mathfrak{h}))\cup\cdots\cup V(\mathfrak{h}-\chi_p(\mathfrak{h}))$$

for some algebraic $\mathfrak{g}\subset\mathfrak{h}\subset\mathfrak{t}$, and appropriate $\chi_1,\dots,\chi_p$. Then we use Theorem 7.3.1(6) and Proposition 4.5.1. □

Remark 7.4.2. It would be very nice if all $(\chi_i)_{i=1,\dots,p}$ were equivalent under the $\rightarrow$ relation (for $\mathfrak{h}$), so that $B^\chi_{\mathfrak{g}}/J$ would in fact be Morita equivalent to $B^{\chi_1}_{\mathfrak{h}}$. However it is easy to give counterexamples to this.

From the proof of Proposition 7.4.1 we deduce the following corollary :

Corollary 7.4.3. *The Goldie rank of $B^\chi_{\mathfrak{g}}/J(\alpha)$ equals the number of connected components of $\overline{\langle\alpha\rangle}$.*

7.5. **Simplicity.** Given the fact that a ring is simple if and only if its only primitive ideal is the zero ideal, it is possible to deduce from Theorem 7.3.1(6) and Remark 7.3.2 necessary and sufficient conditions for simplicity. However these are somewhat technical. Instead we prove a direct criterion that emphasizes the connection between the simplicity of B^χ and the Cohen-Macaulayness of R^G_χ. We use the same notation as in the previous sections.

Suppose for a moment that $s = 0$, that is $R = k[x_1, \dots, x_r]$, and that $\chi \in X(G)$. Then it was shown by Stanley in [25, Th. 3.2] that if χ is of the form $\sum_{i=1}^r \theta_i\eta_i$ in $X(G)_{\mathbb{Q}}$ with $\theta_i \in]-1, 0]$ then R^G_χ is Cohen-Macaulay. On the other hand a straightforward generalization of [28, Theorem 6.2.5] shows that if B^χ is simple, then R^G_χ is Cohen-Macaulay.

So it is not unreasonable to suppose that there is a connection between the condition $\chi = \sum_{i=1}^r \theta_i\eta_i$, $\theta_i \in]-1, 0]$ and the simplicity of B^χ. Corollary 7.5.2 below goes in this direction.

To state the result we assume that s and χ are general again. That is $R = k[x_1, \dots, x_r, x_{r+1}^{\pm 1}, \dots, x_{r+s}^{\pm 1}]$ and $\chi \in \mathfrak{g}^*$.

Put $t = \dim \mathfrak{g}$ and choose an identification $X(G) \cong \mathbb{Z}^t$. So there is a corresponding identification of $\mathfrak{g}^*$ with k^t. Choose furthermore a $\mathbb{Q}$-linear projection $\mathrm{pr} : k \to \mathbb{Q}$ and denote with the same symbol the corresponding projection $\mathfrak{g}^* \to \mathbb{Q}^t$. If $\chi \in \mathfrak{g}^*$ and $\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ then clearly $\mathrm{pr}(\chi) = \sum_i \mathrm{pr}(\alpha_i)\eta_i$.

Let K be a maximal subset of $\{1, \dots, n\}$ such that there exist $(\mu_i)_{\in K} \in \mathbb{Q}$ different from zero, with $\sum_{i\in K} \mu_i\eta_i = 0$. It is easy to see that such a K is unique.

The following proposition gives a somewhat technical simplicity criterion for B^χ. The main applications are corollary 7.5.2 and Proposition 7.6.3 below.

Proposition 7.5.1. *Assume that* $\mathrm{pr}(\chi)$ *is of the form* $\sum_{i=1}^n \theta_i\eta_i$ *with* $(\theta_i)_i \in \mathbb{Q}$ *and*

$$\theta_i \in]-1, 0[\qquad \text{for } i \in K \cap \{1, \dots, r\} \tag{7.16}$$

Then B^χ *is simple.*

Proof. According to Proposition 3.3.1 and Theorem 7.3.1(1) it is sufficient to show that for all $\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ it is true that $\overline{\langle\alpha\rangle}_{B^\chi} = V(\mathfrak{g} - \chi(\mathfrak{g}))$. One has

$$\langle\alpha\rangle_{B^\chi} = \left\{\beta \in V(\mathfrak{g} - \chi(\mathfrak{g})) \mid \beta \cong \alpha \mod \mathbb{Z}^n \quad \text{and} \quad \forall i \in T : \begin{array}{l} \alpha_i \geq 0 \Rightarrow \beta_i \geq 0 \\ \alpha_i < 0 \Rightarrow \beta_i < 0 \end{array}\right\}$$

(recall that $T = \{1, \dots, r\} \cap \{i \mid \alpha_i \in \mathbb{Z}\}$). Since $\langle\alpha\rangle - \alpha$ is of the form $C \cap L$, as in Proposition 7.1.2, we know by [28, Lemma 3.4] that $\langle\alpha\rangle_{B^\chi}$ is Zariski dense iff there exists $\epsilon \in \mathbb{Q}^n$ such that $\sum \epsilon_i\eta_i = 0$ and

$$\forall i \in \{1, \dots, r\} \cap K, \alpha_i \in \mathbb{Z} : \begin{array}{l} \alpha_i \geq 0 \Rightarrow \epsilon_i > 0 \\ \alpha_i < 0 \Rightarrow \epsilon_i < 0 \end{array} \tag{7.17}$$

(the restriction to $i \in K$ is due to the fact that in [28, lemma 3.4] the λ_i that describe C are, implicitly, assumed to be non-zero).

Suppose that $\mathrm{pr}(\chi) = \sum_{i=1}^n \theta_i\eta_i$ where $(\theta_i)_i$ satisfies (7.16). Then $\epsilon = \mathrm{pr}(\alpha) - \theta$ obviously satisfies (7.17). $\square$

Corollary 7.5.2. (1) *If* 0 *is in the relative interior of the convex polyhedral cone spanned by the weights* $(\eta_i)_{i\in\{1,\dots,r\}}$, $(\pm\eta_i)_{i\in\{r+1,\dots,r+s\}}$ *then the conclusion of Proposition 7.5.1 remains valid if we replace* (7.16) *by*

$$\theta_i \in]-1,0] \qquad \text{for } i \in \{1,\dots,r\}$$

(2) *If* 0 *is in the relative interior of the convex polyhedral cone spanned by the weights* $(\eta_i)_{i\in\{1,\dots,r\}}$, $(\pm\eta_i)_{i\in\{r+1,\dots,r+s\}}$ *then* B^{triv} *is simple ("triv" is the trivial character 0).*

Proof. (1) The hypotheses imply that there exist $(\delta_i)_i \in \mathbb{Q}^n$, $\forall i \in \{1,\dots,r\}$: $\delta_i > 0$ such that $\sum_{i=1}^n \delta_i\eta_i = 0$. If we replace θ with $\theta - \mu\delta$ with $\mu \in \mathbb{Q}^+$, sufficiently small, but positive, we obtain that (7.16) is fulfilled.

(2) This is obvious from (1). □

Corollary 7.5.3. *Assume that* G *is an abelian linear algebraic group acting rationally on a smooth affine variety. Then the ring of differential operators* $D(X/\!\!/G)$ *is simple.*

Proof. By the Luna Slice Theorem we can reduce to the case where X is a representation [31]. Furthermore by [22, Thm 10.6] (see also [18, Prop. 3.7]) we can further reduce to $X = k^r \times (k^*)^s$, G acting diagonally with trivial principal isotropy groups (TPIG) and such that $D(X/\!\!/G) = D(X)^G/(\mathfrak{g} - \chi(\mathfrak{g}))$. Let G° be the connected component of G. The fact that X has TPIG implies that corollary 7.5.2(2) holds and thus $D(X)^{G^\circ}/(\mathfrak{g} - \chi(\mathfrak{g}))$ is simple. Furthermore $H = G/G^\circ$ is a finite group and using again the fact that X has TPIG yields that H acts faithfully on $X/\!\!/G^\circ$. Thus if we filter $D(X)^{G^\circ}/(\mathfrak{g} - \chi(\mathfrak{g}))$ by order of differential operators then H also acts faithfully on the associated graded ring. So H acts by outer automorphisms and hence by [17, cor. 2.6] $D(X/\!\!/G) = D(X)^G/(\mathfrak{g} - \chi(\mathfrak{g})) = (D(X)^{G^\circ}/(\mathfrak{g} - \chi(\mathfrak{g})))^H$ is simple. □

7.6. **Simplicity and the $\to$-relation.** We will first investigate when $\chi, \chi' \in \mathfrak{g}^*$ are comparable. Let $K \subset \{1,\dots,n\}$ be as in the previous section.

Proposition 7.6.1. *If* $\chi - \chi' \in \sum_{i\in K} \mathbb{Z}\eta_i$ *then* χ, χ' *are comparable.*

Proof. According to Proposition 4.4.2 we have to show that $\chi + \eta_i$ and χ are comparable if $i \in K$.

Since $x_i \in A^{\mathfrak{g}}_{\eta_i}$, $\partial_i \in A^{\mathfrak{g}}_{-\eta_i}$ it suffices to show that $\partial_i x_i \notin (\mathfrak{g} - \chi(\mathfrak{g}))$. For this it suffices that $\pi_i \notin \mathfrak{g}$.

So suppose on the contrary that $\pi_i \in \mathfrak{g}$ and let $\sum_{i\in K} \mu_i\eta_i = 0$, $\mu_i \in \mathbb{Q}$, $\mu_i \neq 0$. Now η_i is the composition of the inclusion $\mathfrak{g} \to \mathfrak{t}$ and the projection on the i'th factor $\mathfrak{t} \to k$. Hence $\eta_j(\pi_i) = \delta_{ij}$. Evaluating $\sum_{i\in K} \mu_i\eta_i$ on π_i yields $\mu_i = 0$, contradicting the choice of the μ's. □

Lemma 7.6.2. *Let* $\operatorname{pr} : k \to \mathbb{Q}$ *be as in §7.5. Let* $\chi \in \mathfrak{g}^*$. *Then there always exists* $\chi' \in \mathfrak{g}^*$ *such that* $\chi' \cong \chi \bmod \sum_{i\in K} \mathbb{Z}\eta_i$ *and such that*

$$\operatorname{pr}(\chi') = \sum_{i=1}^n \theta_i\eta_i$$

with $\theta_i \in]-1,0[\cap \mathbb{Q}$ *for all* $i \in K$.

Proof. Let $\sum_{i\in K}\mu_i\eta_i = 0$, $\mu_i \in \mathbb{Q}$ different from 0. Write $\chi = \sum_{i=1}^n z_i\eta_i$, $z_i \in k$. By replacing z with $z+\epsilon\mu$, $\epsilon \in \mathbb{Q}$ chosen suitably, we may assume that $\mathrm{pr}(z_i) \notin \mathbb{Z}$ for $i \in K$.

Then we write $\mathrm{pr}(z_i) = n_i + \theta_i$, $n_i \in \mathbb{Z}$, $\theta_i \in]-1,0[$ and we put $\chi' = \sum_{i=1}^n (z_i - n_i)\eta_i$. It is clear that χ' has the required properties. □

Now let us call $\chi \in \mathfrak{g}^*$ minimal if for all $\chi' \in \mathfrak{g}^*$ with $\chi \to \chi'$ one has $\chi' \to \chi$ (we think of $\to$ as $\geq$).

The following is the main result of this section.

Proposition 7.6.3. *Let $\chi \in \mathfrak{g}^*$. Then B^χ is simple if and only if χ is minimal.*

Proof. Assume first that B^χ is simple and $\chi \to \chi'$. By Prop. 4.4.2, χ and χ' are comparable. Hence in particular $B^{\chi,\chi'}B^{\chi',\chi}$ is a non-zero ideal in B^χ. Since B^χ is simple this implies $B^{\chi,\chi'}B^{\chi',\chi} = B^\chi$ and hence $\chi' \to \chi$ which is what we had to show.

Conversely assume that χ is minimal. By lemma 7.6.2 there exist χ' comparable to χ such that $B^{\chi'}$ is simple. So $B^{\chi',\chi}B^{\chi,\chi'} = B^{\chi'}$ and thus $\chi \to \chi'$. Since χ is minimal this implies $\chi' \to \chi$ and hence B^χ and $B^{\chi'}$ are Morita equivalent. Thus B^χ is also simple. □

7.7. **Primitive ideals and the $\to$-relation.** We will consider pairs (ψ,θ) satisfying (1)(2)(3) of definition 7.2.1 and having the additional property that θ is in the image of

$$\sum_{\substack{i\in\{1,\dots,r\}\\ \langle\psi,\eta_i\rangle=0}} (k-\mathbb{Z})\eta_i + \sum_{i\notin\{1,\dots,r\}} k\eta_i$$

We will call two such pairs (ψ,θ), (ψ',θ') equivalent if

$$\begin{aligned}\{i \mid \langle\psi',\eta_i\rangle < 0\} &= \{i \mid \langle\psi,\eta_i\rangle < 0\}\\ \{i \mid \langle\psi',\eta_i\rangle > 0\} &= \{i \mid \langle\psi,\eta_i\rangle > 0\}\\ \theta' \cong \theta \quad &\mathrm{mod} \sum_{\langle\psi,\eta_i\rangle=0} \mathbb{Z}\eta_i\end{aligned}$$

and we will denote the set of all equivalence classes by $\mathcal{P}$.

We define $(\psi,\theta) \geq (\psi',\theta')$ when

$$\begin{aligned}\{i \mid \langle\psi',\eta_i\rangle < 0\} &\subset \{i \mid \langle\psi,\eta_i\rangle < 0\}\\ \{i \mid \langle\psi',\eta_i\rangle > 0\} &\subset \{i \mid \langle\psi,\eta_i\rangle > 0\}\\ \theta' \cong \theta \quad &\mathrm{mod} \sum_{\langle\psi',\eta_i\rangle=0} \mathbb{Z}\eta_i\end{aligned}$$

and in this way $\mathcal{P}$ becomes a partially ordered set. Note that the order relation on $\mathcal{P}$ corresponds to the inclusions between the $S_{\psi,\theta}$ as given by Proposition 7.2.5.

If $\chi \in \mathfrak{g}^*$ then we define $\mathcal{P}_\chi$ as the set of all equivalence classes of pairs (ψ,θ) attached to χ. If $(\psi,\theta) \in \mathcal{P}_\chi$ then we write $J(\psi,\theta)_{B^\chi}$ for the primitive ideal associated to $S_{\psi,\theta}$.

In this section we will prove the following result.

Proposition 7.7.1. (1) $\bigcup_{\chi\in\mathfrak{g}^*}\mathcal{P}_\chi = \mathcal{P}$

(2) *The map $\mathcal{P}_\chi \to \mathrm{Prim}(B_\chi) : (\psi,\theta) \mapsto J(\psi,\theta)_{B^\chi}$ is an order preserving bijection.*

(3) $\chi' \to \chi$ *if and only if* $\mathcal{P}_\chi \subset \mathcal{P}_{\chi'}$.

The proof of this result is partially based upon the following proposition which shows that $\mathcal{P}$ may be considered as the set of all primitive ideals in all B^χ, subject to a natural identification.

If J is an ideal of B^χ we set as in §4.4

$$\tilde{J} = \{x \in B^{\chi'} \mid B^{\chi,\chi'} x B^{\chi',\chi} \subset J\}$$

Proposition 7.7.2. *Assume that* (ψ, θ) *is attached to* χ. *Then*

$$\widetilde{J(\psi,\theta)}_{B^\chi} = \begin{cases} J(\psi,\theta)_{B^{\chi'}} & \text{if } (\psi,\theta) \text{ is attached to } \chi' \\ B^{\chi'} & \text{otherwise} \end{cases}$$

Proof. Let $\beta \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ be as in definition 7.2.1(4)(5). According to Proposition 4.4.5 we have to show that

$$\langle\beta\rangle_A \cap V(\mathfrak{g} - \chi'(\mathfrak{g})) \neq \emptyset \quad \text{iff} \quad (\psi,\theta) \text{ is attached to } \chi'$$

Assume first that there exist $\beta' \in \langle\beta\rangle_A \cap V(\mathfrak{g} - \chi'(\mathfrak{g}))$. Then corollary 6.2 implies that β' satisfies definition 7.2.1(4)(5) and hence (ψ,θ) is attached to χ'.

Conversely assume that (ψ,θ) is attached to χ' and let $\beta' \in V(\mathfrak{g} - \chi'(\mathfrak{g}))$ be as in definition 7.2.1(4)(5). According to 7.2.1(4) there exist $(u_i)_{\langle\psi,\eta_i\rangle=0} \in \mathbb{Z}$ such that $\sum_{\langle\psi,\eta_i\rangle=0}(\beta_i - \beta'_i - u_i)\eta_i = 0$. Put $\phi_i = \beta_i - \beta'_i - u_i$ if $\langle\psi,\eta_i\rangle = 0$ and $\phi_i = 0$ otherwise. We replace β' by $\beta' + \phi$. Then this new β' still satisfies 7.2.1(4)(5) but also $\beta \cong \beta' \mod \mathbb{Z}^n$. Hence corollary 6.2 yields that $\beta' \in \langle\beta\rangle_A \cap V(\mathfrak{g} - \chi'(\mathfrak{g}))$. □

Proof of Proposition 7.7.1. (1) is clear. (2) follows from Theorem 7.3.1(4) and Propositions 7.2.4 and 7.2.5. So we only have to prove (3).

According to Proposition 4.4.5, Theorem 4.4.4 and Proposition 7.7.2 we have the following chain of equivalences.

$$\begin{aligned} \chi' \to \chi \quad & \text{iff} \quad \forall \alpha \in V(\mathfrak{g} - \chi(\mathfrak{g})) : \widetilde{J(\alpha)}_{B^\chi} \neq B^{\chi'} \\ & \text{iff} \quad \forall (\psi,\theta) \in \mathcal{P}_\chi : \widetilde{J(\psi,\theta)}_{B^\chi} \neq B^{\chi'} \\ & \text{iff} \quad \forall (\psi,\theta) \in \mathcal{P}_\chi : (\psi,\theta) \in \mathcal{P}_{\chi'} \end{aligned}$$

This proves the proposition. □

8. Dimension theory for B^χ

In this section we keep the notations of §6,§7. Our aim in this section is to give the values of some classical dimensions for B^χ. For completeness we also restate some results already proved in [18]. The case of global dimension is treated in section §9.

8.1. **Krull dimension.** The Krull dimension of B^χ is rather easy to compute. One uses the the following lemma.

Lemma 8.1.1. *Let S be a ring graded by a group G, H a subgroup of G and let B be the ring obtained from A by taking the graded components corresponding to H. Then*

$$\mathrm{Kdim}(B) \leq \mathrm{Kdim}(A)$$

Proof. It is easy to see that [15, 6.5.3 (i)] applies. □

Theorem 8.1.2. $\operatorname{Kdim}(B^\chi) = \operatorname{Kdim}(B^\chi)_0 = \dim \mathfrak{t} - \dim \mathfrak{g}$

Proof. We have $\operatorname{Kdim}(B^\chi) \geq \operatorname{Kdim}(B^\chi)_0$ because of the above lemma. On the other hand we have by definition

$$B^\chi = A^{\mathfrak{g}}/(\mathfrak{g} - \chi(\mathfrak{g}))$$

and $\mathfrak{g} - \chi(\mathfrak{g})$ is a regular sequence in $A^{\mathfrak{g}}$. So by the lemma and by [15, 6.3.9] we obtain

$$\operatorname{Kdim}(B^\chi) \leq \operatorname{Kdim} A^{\mathfrak{g}} - \dim \mathfrak{g} \leq \operatorname{Kdim} A - \dim \mathfrak{g} = \dim \mathfrak{t} - \dim \mathfrak{g}$$

Here we have used that $\operatorname{Kdim} A = \dim \mathfrak{t}$ [15, Thm. 6.6.15]. $\square$

8.2. **GK-dimension.** To study the GK-dimension of B^χ and its modules we filter B^χ by order of differential operators. We start by filtering $A = R[\partial_1, \dots, \partial_n]$ by the degree of the ∂'s. That is

$$F_m A = \left\{ \sum a_{(u)} \partial_1^{u_1} \cdots \partial_n^{u_n} \mid \sum u_i \leq m \right\}$$

and one has $\operatorname{gr}_F A = R[\bar\partial_1, \dots, \bar\partial_n]$ which is a polynomial ring over R.

This filtration induces a filtration on A^G which we also denote by F. Since F is G-invariant and G is reductive we have $\operatorname{gr}_F(A^G) = (\operatorname{gr} A)^G$.

The filtration F on A^G induces a filtration on $B^\chi = A^{\mathfrak{g}}/(\mathfrak{g} - \chi(\mathfrak{g}))$ and since $\mathfrak{g}$ is generated by a regular sequence in $(\operatorname{gr}_F A)^{\mathfrak{g}}$ we easily deduce that

$$\operatorname{gr}_F B^\chi = (\operatorname{gr}_F A)^G/(\mathfrak{g}) \tag{8.1}$$

If M is a finitely generated A^G-module then there always exists a so-called "good" filtration on M. That is a filtration $(F_m M)_m$ such that $\operatorname{gr}_F M$ is finitely generated as $\operatorname{gr}_F A^G$-module. For such a filtration it follows from [16, §1.4] that

$$\operatorname{GKdim}_{A^G} M = \operatorname{GKdim}_{\operatorname{gr}_F A^G} \operatorname{gr}_F M \tag{8.2}$$

Theorem 8.2.1.
1. *The GK-dimension of an A^G-module is either an integer or infinite.*
2. *GK-dimension is exact for A^G-modules (see* [11, Chapter 5] *for the definition of exactness).*
3. $\operatorname{GKdim} A^G = 2n - \dim G$.
4. $\operatorname{GKdim} B^\chi = 2(n - \dim G)$

Proof. (1) and (2) follow from the discussion above. (3) and (4) are restatements of [18, Cor. 3.2]. $\square$

Corollary 8.2.2. *Let $\alpha \in \mathfrak{t}^*$. Then*

$$\operatorname{GKdim} B^\chi/J(\alpha) = 2 \operatorname{GKdim}(B^\chi/J(\alpha))_0 = 2 \dim \overline{\langle\alpha\rangle}_{B^\chi} \tag{8.3}$$

Proof. Since the last equality in (8.3) is a tautology we concentrate on the first one. According to Proposition 7.2.4 there exists an algebraic $\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{t}$ and $\chi_1, \dots, \chi_p \in \mathfrak{h}^*$ such that

$$B^\chi/J(\alpha) = \begin{pmatrix} B_{\mathfrak{h}}^{\chi_1} & B_{\mathfrak{h}}^{\chi_1,\chi_2} & & \\ B_{\mathfrak{h}}^{\chi_2,\chi_1} & B_{\mathfrak{h}}^{\chi_1} & & \\ & & \ddots & \\ & & & B_{\mathfrak{h}}^{\chi_p} \end{pmatrix} \tag{8.4}$$

Let $S = \oplus B_{\mathfrak{h}}^{\chi_i}$ be embedded diagonally in the right-hand side of (8.4). Then $B^\chi/J(\alpha)$ is a finitely generated as a module over S and hence by [11, Prop. 5.5]

$$\operatorname{GKdim} B^\chi/J(\alpha) = \operatorname{GKdim} S = \max_i \operatorname{GKdim} B_{\mathfrak{h}}^{\chi_i}$$

By the same argument we have

$$\operatorname{GKdim}(B^\chi/J(\alpha))_0 = \operatorname{GKdim} S_0 = \max_i \operatorname{GKdim}(B_{\mathfrak{h}}^{\chi_i})_0$$

and hence to prove the first equality in (8.3) we may assume that $J(\alpha) = 0$. But then we may invoke Theorem 8.2.1. □

Our next aim is to compute the GK-dimension of objects in $\mathcal{O}_{B^\chi}^{(\infty)}$.

Proposition 8.2.3. *Let $\alpha \in V(\mathfrak{g})$. Then*

$$\operatorname{GKdim}_{B^\chi} L(\alpha)_{B^\chi} = \dim \overline{\langle\alpha\rangle}_{B_\chi}$$

To give the proof of this proposition we have to make a few preparations.

Let $E, \| \ \|$ be a normed finite dimensional vector space over $\mathbb{R}$ and let M be an E-graded k-vector-space. Then we define for $z \in \mathbb{R}$

$$d_{M,\| \ \|}(z) = \sum_{\|x\| \leq z} \dim_k M_x$$

and

$$\operatorname{GKdim}(M, E) = \limsup_{n \in \mathbb{N}} \frac{\log d_{M,\| \ \|}(n)}{\log n}$$

Since all norms on E are equivalent $\operatorname{GKdim}(M, E)$ does not depend upon the choice of $\| \ \|$.

Theorem 8.2.4. (1) *Let E be a finite dimensional vector space over $\mathbb{R}$ and let A, M be respectively an E-graded k-algebra and an E-graded A-module. Then*

$$\operatorname{GKdim}_A M \leq \operatorname{GKdim}(M, E)$$

(2) *Assume now in addition that A is commutative, A, M are finitely generated, A is graded by some lattice in E and $\dim M_x \in \{0, 1\}$ for all $x \in E$. Then*

$$\operatorname{GKdim}_A M = \operatorname{GKdim}(M, E)$$

Proof. (1) Let V be a finite dimensional subspace of A and F a finite dimensional subspace of M. By enlarging V and F if necessary we may assume that V and F are graded and $1 \in V$.

As in [11] we put $d_{V,F}(n) = \dim V^n F$. Choose a norm $\| \ \|$ on E and put

$$s = \max_{u \in \operatorname{Supp} V} \|u\|$$
$$t = \max_{u \in \operatorname{Supp} F} \|u\|$$

Then $V^n F \subset \oplus_{\|u\| \leq ns+t} M_u$ so $d_{V,F}(n) \leq d_{M,\| \ \|}(ns + t)$ and thus

$$\limsup_n \frac{\log d_{V,F}(n)}{\log n} \leq \operatorname{GKdim}(M, E)$$

Since this holds for all V, F we obtain (1).

(2) We let V, F be as in (1) but we now assume that they generate A and M. On the $\mathbb{R}$-vector-space $\{(x_l)_{l\in \operatorname{Supp} V} \mid x_l \in \mathbb{R}\}$ we define a norm by $\|x\| = \sum_l |x_l|$.

It is clear that we now have the following

$$(V^n F)_u = \sum_{\substack{\|N\|\le n\\ e\in \operatorname{Supp} F}} \left(\prod_{l\in \operatorname{Supp} V} V_l^{N_l}\right) F_e$$

where the sum is restricted to the positive solutions N of

$$\sum_{l\in \operatorname{Supp} V} N_l l + e = u$$

By lemma 8.2.6 below we know that there exist α, β such that if $\|N\| > \alpha\|u-e\| + \beta$ then there exists a solution N' to

$$\sum_{l\in \operatorname{Supp} V} N'_l l = 0$$

such that $N' \le N$ and $\|N - N'\| \le \alpha\|u-e\| + \beta$. Note that, to be able to apply this lemma, we have used that $\operatorname{Supp} V \subset \operatorname{Supp} A$ is contained in a lattice in E.

For such an N' we have that

$$\prod_{l\in \operatorname{Supp} V} V_l^{N'_l} \subset A_0$$

Hence if we define

$$\beta' = \alpha\left(\max_{e\in \operatorname{Supp} F} \|e\|\right) + \beta$$

then we have that $F^{(u)} = (V^{\lfloor \alpha\|u\|+\beta'\rfloor} F)_u$ is a generating vector-space for M_u as A_0-module. Since $\dim M_u = 0, 1$ this yields that

$$\dim(V^n F)_u = 1$$

if $n \ge \alpha\|u\| + \beta'$ and $u \in \operatorname{Supp} M$. So

$$\begin{aligned} d_{V,F}(n) &\ge \#\left\{u \in \operatorname{Supp} M \mid \|u\| \le \frac{n-\beta'}{\alpha}\right\} \\ &= d_{M,\|\ \|}\left(\frac{n-\beta'}{\alpha}\right) \end{aligned}$$

which yields

$$\operatorname{GKdim}_A M \ge \operatorname{GKdim}(M, E)$$

Since the reverse inequality was proved in (1) we are done. $\square$

Proof of Proposition 8.2.3. Let us write $L(\alpha)$ for $L(\alpha)_{B^\chi}$. We may clearly compute the GK-dimension with respect to A^G. Then we may use the filtration F on A^G which was defined earlier. This filtration is compatible with the $\mathfrak{t}^*$-grading on A^G. Furthermore by inspection of the proof of [11, lemma 6.7] yields that it is possible to make $L(\alpha)$ in to a filtered A^G module such that

- $\operatorname{gr}_F L(\alpha)$ is a finitely generated $\operatorname{gr}_F A^G$-module.
- All $F_m L(\alpha)$ are $\mathfrak{t}^*$-graded subspaces of $L(\alpha)$.

Using (8.2) and Theorem 8.2.4(2) we find that

$$\operatorname{GKdim}_{A^G} L(\alpha) = \operatorname{GKdim}_{\operatorname{gr}(A^G)} \operatorname{gr} L(\alpha) = \operatorname{GKdim}(L(\alpha), V(\mathfrak{g}))$$

Now according to Proposition 7.1.2 and the discussion thereafter, there exist an algebraic $\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{t}$ such that $\operatorname{Supp} L(\alpha)$ is a union of dense cones in a finite number of translates of $V(\mathfrak{h})$. But then it is easy to see that

$$\operatorname{GKdim}(L(\alpha), V(\mathfrak{g})) = \dim V(\mathfrak{h}) = \dim \overline{\langle\alpha\rangle}_{B^\chi} \quad \square$$

Corollary 8.2.5. *Assume that $M \in \mathcal{O}_{B^\chi}^{(\infty)}$ is finitely generated. Then*

$$\operatorname{GKdim} M = \frac{1}{2} \operatorname{GKdim}(B^\chi / \operatorname{Ann} M)$$

Proof. Let $L(\alpha_1), \ldots, L(\alpha_p)$ be the Jordan-Holder quotients of M (with multiplicity). Clearly $\operatorname{GKdim} M = \max \operatorname{GKdim} L(\alpha_i)$. Let t be such that $\operatorname{GKdim} L(\alpha_t)$ is maximal.

If we put $\alpha_1, \ldots, \alpha_p$ in the correct order then

$$J(\alpha_1) \cdots J(\alpha_p) \subset \operatorname{Ann} M \subset J(\alpha_t)$$

So

$$\operatorname{GKdim}(B^\chi / J(\alpha_1) \cdots J(\alpha_p)) \geq \operatorname{GKdim}(B^\chi / \operatorname{Ann} M) \geq \operatorname{GKdim}(B^\chi / J(\alpha_t)) \tag{8.5}$$

Now since $\operatorname{rad}(J(\alpha_1) \cdots J(\alpha_t)) = J(\alpha_1) \cap \cdots J(\alpha_p)$ we obtain by [11, Prop. 5.7] that

$$\operatorname{GKdim}(B^\chi / J(\alpha_1) \cdots J(\alpha_p)) = \operatorname{GKdim}(B^\chi / J(\alpha_s))$$

for some s. Using Proposition 8.2.3 and (8.5) we now obtain

$$2 \operatorname{GKdim} L(\alpha_s) \geq \operatorname{GKdim} B^\chi / \operatorname{Ann} M \geq 2 \operatorname{GKdim} L(\alpha_t)$$

Hence by the choice of t

$$\operatorname{GKdim}(B^\chi / \operatorname{Ann} M) = 2 \operatorname{GKdim} L(\alpha_t) = 2 \operatorname{GKdim} M \quad \square$$

In the proof of Theorem 8.2.4 we used lemma 8.2.6 below. This lemma has perhaps some independent interest.

Assume that ϕ is an $m \times n$-matrix over $\mathbb{Z}$. If $x, y \in \mathbb{N}^n$ then we say that $x < y$ if $x_i \leq y_i$ for all i and $x_i < y_i$ for at least one i. We will call $y \in \mathbb{N}^n$ minimal with respect to ϕ if there does not exist $x \in \mathbb{N}^n$ such that $\phi x = \phi y$ and $x < y$.

Lemma 8.2.6. *Choose norms $\|\ \|$ on $\mathbb{R}^m$, $\mathbb{R}^n$. Then there exist constants α, β such that for all minimal y with respect to ϕ one has*

$$\|y\| \leq \alpha \|\phi y\| + \beta$$

Proof. Since all norms on $\mathbb{R}^m$ and $\mathbb{R}^n$ are equivalent we may choose specific ones. We take the euclidean norm on $\mathbb{R}^m$ and on $\mathbb{R}^n$ we take $\|x\| = \sum_i |x_i|$.

We now proceed by translating our problem into one for torus invariants. Let $T = (\mathbb{C}^*)^m$, $R = \mathbb{C}[u_1, \ldots, u_n]$. We consider R as a $\mathbb{Z}^n$-graded ring in the obvious way. Define $\eta_i \in X(T)$ by

$$\eta_i(z_1, \ldots, z_m) = z_1^{\phi_{1i}} \cdots z_m^{\phi_{mi}}$$

We let T act on u_i with weight η_i. This defines a T action on R. As usual, for a graded object X we let $\operatorname{Supp} X$ stand for $\{\alpha \mid X_\alpha \neq 0\}$. The solutions of $\phi y = 0$ correspond to $\operatorname{Supp} R^T$ and the y that are minimal with respect to ϕ are given by

$\operatorname{Supp} R/(R^T_{>0}R)$ where $R^T_{>0}$ is the irrelevant ideal of R^T (considered as a positively graded ring in the natural way).

Let $I = \operatorname{rad}(R^T_{>0}R)$, $\bar{R} = R/(R^T_{>0}R)$ and let $\bar{I}$ be the image of I in R. Then

$$\operatorname{Supp} \bar{R} = \operatorname{Supp} \bar{R}/\bar{I} \cup \operatorname{Supp} \bar{I}/\bar{I}^2 \cup \cdots \quad \text{(finite union)}$$

Since $\bar{I}$ is finitely generated this implies that there exist $x_1, \dots, x_t \in \mathbb{Z}^n$ such that

$$\operatorname{Supp} \bar{R} \subset \bigcup_j (x_j + \operatorname{Supp} R/I) \tag{8.6}$$

Now we describe $\operatorname{Supp} R/I$. Let $X = \operatorname{Spec} R$. For $\lambda \in Y(T)$ a one-parameter subgroup in T one defines

$$X_\lambda = \{x \in X \mid \lim_{t \to 0} \lambda(t)x = 0\}$$

Then the Hilbert-Mumford criterion yields that the irreducible components of $\operatorname{Spec} R/I$ are of the form X_λ for suitable λ.

If $P_1, \dots, P_s$ are the minimal primes of R/I then R/I injects in $\oplus R/P_i$ and hence

$$\operatorname{Supp} R/I \subset \bigcup_i \operatorname{Supp} R/P_i$$

(of course one has equality here).

Furthermore R/P_i is the coordinate ring of some X_λ and one verifies that

$$\operatorname{Supp} R/P_i = \{(a_i)_i \in \mathbb{N}^n \mid \langle \lambda, \eta_i \rangle \geq 0 \Rightarrow a_i = 0\} \tag{8.7}$$

Since $T = (\mathbb{C}^*)^m$ there are canonical identifications $X(T)_\mathbb{R} = Y(T)_\mathbb{R} = \mathbb{R}^m$ and we use these to put the euclidean norm on $X(T)_\mathbb{R}$ and $Y(T)_\mathbb{R}$.

Let $a = (a_i)_i \in \operatorname{Supp} R/P_i$ and put $\zeta = \sum_i a_i \eta_i$. Then $\sum_i a_i \langle \lambda, \eta_i \rangle = \langle \lambda, \zeta \rangle$. Then (8.7) implies that

$$\|a\| = \sum a_i \leq \frac{|\langle \lambda, \zeta \rangle|}{\min_{\langle \lambda, \eta_i \rangle < 0} |\langle \lambda, \eta_i \rangle|} \leq \frac{\|\lambda\|}{\min_{\langle \lambda, \eta_i \rangle < 0} |\langle \lambda, \eta_i \rangle|} \|\zeta\|$$

We now take α to be the maximum of all

$$\frac{\|\lambda\|}{\min_{\langle \lambda, \eta_i \rangle < 0} |\langle \lambda, \eta_i \rangle|}$$

where the λ's are taken such that X_λ runs over all irreducible components of $\operatorname{Spec} R/I$.

Furthermore we put

$$\beta = \max_j (\|x_j\| + \alpha \|\sum_i x_{ji} \eta_i\|)$$

If $y \in \operatorname{Supp} \bar{R}$ then we may write

$$y = x_j + a$$

for some j and $a \in \operatorname{Supp} R/P_i$ for some i. Hence

$$\begin{aligned}
\|y\| &\le \|x_j\| + \|a\| \\
&\le \|x_j\| + \alpha \|\sum_i a_i \eta_i\| \\
&\le \|x_j\| + \alpha \|\sum_i (a_i + x_{ji})\eta_i\| + \alpha \|\sum_i x_{ji}\eta_i\| \\
&\le \beta + \alpha \|\sum_i y_i \eta_i\|
\end{aligned}$$

Translating this back to the setting of the statement of the lemma yields the desired result. □

8.3. **GK-dimension of annihilators.** If U is the enveloping algebra of an algebraic Lie algebra then a famous result of Gabber and Joseph [11, Thm 9.11] asserts that for any finitely generated U-module M

$$2 \operatorname{GKdim} M \ge \operatorname{GKdim}(U/\operatorname{Ann} M) \tag{8.8}$$

This result was used by Levasseur for the computation of the injective dimensions of minimal primitive quotients of enveloping algebras (see [12]).

Our aim in this section is to prove a result similar to (8.8) for rings of differential operators on torus invariants. Unfortunately we have only been able to generalize (8.8) in the case that M is simple. However this is sufficient to generalize the proof of Levasseur to B^χ.

We revert to the notation which has been in use in sections 6,7. As indicated in the previous paragraph we will prove the following result.

Theorem 8.3.1. *Let M be a simple B^χ-module. Then*

$$2 \operatorname{GKdim} M \ge \operatorname{GKdim}(B^\chi/\operatorname{Ann} M)$$

Proof. For simplicity we write $B = B^\chi$ and we let $\mathfrak{s}$ stand for the image of $\mathfrak{t}$ in B_0. Thus B_0 is the symmetric algebra of $\mathfrak{s}$.

Put $J = \operatorname{Ann} M$. Since M is simple J is a primitive ideal. Therefore by Theorem 7.4.1 there exists an algebraic $\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{t}$ such that

$$B/J = \begin{pmatrix} B^1 & B^{1,2} & & \\ B^{2,1} & B^2 & & \\ & & \ddots & \\ & & & B^p \end{pmatrix} \tag{8.9}$$

where $B^i = B_{\mathfrak{h}}^{\chi_i}$, $B^{i,j} = B_{\mathfrak{h}}^{\chi_i,\chi_j}$. Let $e_i \in B/J$ be the idempotent which corresponds in the right-hand side of (8.9) to the matrix which has 1 in location (i,i) and zero elsewhere. Then it is easy to verify the following

- all $e_i M$ are either zero or simple B^i-modules;
- $\operatorname{Ann}_{B^i} e_i M = 0$.

Now let $S = \oplus_i B^i$ be embedded diagonally in the right-hand side of (8.9). Then B/J is finitely generated as a module over S, and hence by an extension of [11, Prop 5.5]

$$\operatorname{GKdim}_B M = \operatorname{GKdim}_{B/J} M = \operatorname{GKdim}_S M = \max_i \operatorname{GKdim}_{B^i} e_i M$$

Since we also have by [11, Prop 5.5]

$$\operatorname{GKdim} B/J = \operatorname{GKdim} S = \max_i \operatorname{GKdim} B^i \tag{8.10}$$

we may suppose that $J = \operatorname{Ann}_B M = 0$. (Of course in the maximum that is taken in (8.10), all $\operatorname{GKdim} B^i$ are equal by Theorem 8.2.1.)

So now we suppose that $\operatorname{Ann}_B M = 0$ and we have to show that $\operatorname{GKdim} M \geq \frac{1}{2} \operatorname{GKdim} B = \operatorname{GKdim} B_0$. Hence we will assume on the contrary that $\operatorname{GKdim} M < \operatorname{GKdim} B_0$ and we will obtain a contradiction by showing that necessarily $\operatorname{Ann}_B M \neq 0$.

Choose a non-zero element $x \in M$ in such a way that $P = \operatorname{Ann}_{B_0} x$ has the following properties

(1) $\operatorname{GKdim} B_0/P$ is minimal;
(2) among those P which satisfy (1), P is maximal.

It is an easy exercise that P is prime.

If $\alpha \in \mathfrak{s}^*$ then we denote by $\tau_\alpha : B_0 \to B_0$ the map which sends $\pi \in \mathfrak{s}$ to $\pi - \alpha(\pi)$. Define

$$M_\alpha = \{m \in M \mid t_\alpha(P)m = 0\}$$

It follows from (1) above that if $m \in M_\alpha \setminus \{0\}$ then

$$\operatorname{GKdim} B_0/\operatorname{Ann}_{B_0} m = \operatorname{GKdim} B_0/t_\alpha(P)$$

and since $t_\alpha(P)$ is prime this implies that

$$\operatorname{Ann}_{B_0} m = t_\alpha(P) \tag{8.11}$$

We now make two observations.

Claim 1. Assume that there exist $\alpha_1, \dots, \alpha_p \in \mathfrak{t}^*$ such that

$$(M_{\alpha_1} + \cdots + M_{\alpha_{p-1}}) \cap M_{\alpha_p} \neq 0$$

Then $M_{\alpha_p} = M_{\alpha_i}$ for some $i \in \{1, \dots, p-1\}$.

Proof. To prove this assume that $m_1 + \cdots + m_p = 0$ with $m_i \in M_i$ and $m_p \neq 0$. Then

$$\left(\bigcap_{1 \leq i \leq p-1} \tau_{\alpha_i}(P) \right) m_p = 0$$

and hence by (8.11)

$$\bigcap_{1 \leq i \leq p-1} \tau_{\alpha_i}(P) \subset \tau_{\alpha_p}(P)$$

Thus there must exist an i such that $\tau_{\alpha_i}(P) = \tau_{\alpha_p}(P)$. But then by definition $M_{\alpha_i} = M_{\alpha_p}$. $\square$

Let $L = \operatorname{Supp} B \subset \mathfrak{s}^*$. By construction L is a full sublattice in $\mathfrak{s}^*$.

Claim 2. $M = \sum_{l \in L} M_l$. This follows from the fact that $\sum_{l \in L} M_l$ defines a non-zero submodule of M and M is simple.

Now consider

$$L' = \{l \in L \mid t_l(P) = P\}$$

L' is clearly a sublattice of L, and hence if we put

$$\mathfrak{h} = \bigcap_{l \in L'} \ker l$$

then $\mathfrak{h}$ is algebraic in $\mathfrak{t}$. Furthermore $k \otimes_{\mathbb{Z}} L'$ identifies with $V(\mathfrak{h}) \subset \mathfrak{s}^*$ and since $V(P)$ is $k \otimes_{\mathbb{Z}} L'$-stable we obtain the inequality

$$\dim V(P) \geq \dim V(\mathfrak{h}) \tag{8.12}$$

which will be used below.

If $\alpha \in \mathfrak{h}^*$ then we define

$$M_\alpha = \begin{cases} M_\beta & \text{if } \alpha \in \operatorname{im} L \text{ and } \beta \in L \text{ is a lifting of } \alpha \\ 0 & \text{otherwise} \end{cases}$$

To show that this is well defined let $\beta, \gamma \in L$ be two liftings of $\alpha \in \mathfrak{h}^*$. Then $\beta - \gamma \in L \cap \ker(\mathfrak{s}^* \to \mathfrak{h}^*) = L'$ and hence $\tau_\beta(P) = \tau_\gamma(P)$.

Now we claim that the decomposition $M = \sum_{\alpha \in \mathfrak{h}^*} M_\alpha$ defines as $\mathfrak{h}^*$-grading on M. We only have to show that the sum is direct. To do this, it suffices by claim 1 to show that if $M_\beta = M_\gamma \neq 0$, $\beta, \gamma \in L$ then β, γ have the same image in $\mathfrak{h}^*$. But this is clear since $\tau_\beta(P) = \tau_\gamma(P)$ and $\beta - \gamma \in L$. Hence $\beta - \gamma \in L'$.

So now we have defined a $\mathfrak{h}^*$-grading $M = \bigoplus_{\alpha \in \mathfrak{h}^*} M_\alpha$. Since M is simple, it is easily seen that M_0 is a simple $B^{\mathfrak{h}}$-module. Since $\mathfrak{h} \subset Z(B^{\mathfrak{h}})$ it follows by Quillen's lemma than $\operatorname{Ann}_{B^{\mathfrak{h}}} M_0$ contains $\mathfrak{h} - \zeta(\mathfrak{h})$ for certain $\zeta \in \mathfrak{h}^*$. From the fact that $\operatorname{Ann}_{B^{\mathfrak{h}}} M_0 \cap B_0 = P$ we obtain $\mathfrak{h} - \zeta(\mathfrak{h}) \subset P$. Now the inequality (8.12) yields that in fact $P = (\mathfrak{h} - \zeta(\mathfrak{h}))$.

Now let m be an arbitrary element in M_α, $\alpha \in \mathfrak{h}^*$. Then by definition

$$0 = \tau_\alpha(\mathfrak{h} - \zeta(\mathfrak{h}))m = (h - \zeta(\mathfrak{h}) - \alpha(\mathfrak{h}))m$$

So if we shift the $\mathfrak{h}^*$-grading on M by ζ then we may, and we will, assume that

$$(\mathfrak{h} - \alpha(\mathfrak{h}))M_\alpha = 0 \tag{8.13}$$

We now choose an algebraic $\mathfrak{q}$ in such a way that $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{q}$. Then there is a dual decomposition $\mathfrak{s}^* = \mathfrak{h}^* \oplus \mathfrak{q}^*$ and we define $C = B^{\mathfrak{q}} = \oplus_{l \in \mathfrak{h}^*} B_l$.

Fix $\alpha \in \mathfrak{h}^*$, $m \in M_\alpha$ and put $N = Cm$. N is clearly a $\mathfrak{h}^*$-graded C-submodule of M.

Claim 3. If $\beta \in \mathfrak{h}^*$ is such that $N_\beta \neq 0$ then N_β is a free $B_0/(\mathfrak{h} - \beta(\mathfrak{h}))$-module of rank one.

Proof. Assume that $N_\beta \neq 0$. Clearly $N_\beta = B_{\beta-\alpha}m = B_0 u_{\beta-\alpha} m$. So N_β is generated by 1 element. Since according to (8.13) N_β is annihilated by $(\mathfrak{h} - \beta(\mathfrak{h}))$ we see that N_β is a quotient of $B_0/(\mathfrak{h} - \beta(\mathfrak{h}))$.

On the other hand $N_\beta \subset M_\beta$ which is a torsion free $B_0/(\mathfrak{h} - \beta(\mathfrak{h}))$-module (by the choice of P). This implies that $\operatorname{GKdim}_{B_0} N_\beta = \operatorname{GKdim}_{B_0} M_\beta = \operatorname{GKdim}(B_0/(\mathfrak{h} - \beta(\mathfrak{h}))$. So $N_\beta = B_0/(\mathfrak{h} - \beta(\mathfrak{h}))$. □

Put $C' = C/(\mathfrak{q}) = B^{\mathfrak{q}}/(\mathfrak{q})$. Clearly C' is of the form $B^{\zeta}_{\mathfrak{q}'}$ for suitable $\mathfrak{q}'$, $\zeta \in \mathfrak{q}'^*$. Hence the theory of section 7 applies to C'.

Define

$$Z = \bigcup_{\overline{\langle \alpha \rangle}_{C'} \neq \mathfrak{h}^*} \overline{\langle \alpha \rangle}_{C'} \subset \mathfrak{h}^*$$

By the results in section 7, Z is contained in a finite number of hyperplanes in $\mathfrak{h}^*$.

Claim 4. $\operatorname{Supp} N \subset Z$.

Proof. Put $\overline{N} = B_0/(\mathfrak{q}) \otimes_{B_0} N$. Then we have the following.

- $\operatorname{Supp} \overline{N} = \operatorname{Supp} N$.
- Since $\mathfrak{q}$ is generated by a regular sequence in N

$$\begin{aligned}\operatorname{GKdim} \overline{N} &\leq \operatorname{GKdim} N - \dim \mathfrak{q} \leq \operatorname{GKdim} M - \dim \mathfrak{q} \\ &< \operatorname{GKdim} B_0 - \dim \mathfrak{q} = \dim \mathfrak{s} - \dim \mathfrak{q} = \dim \mathfrak{h}\end{aligned}$$

- By (8.13) is $\overline{N}$ is in $\mathcal{O}^{(1)}_{C'}$. Since $\overline{N}$ is in addition finitely generated, $\overline{N}$ is an extension of a finite number of $L(\alpha)_{C'}$, each of which has GK-dimension less than $\dim \mathfrak{h}$. Thus according to Proposition 8.2.3 we have

$$\operatorname{Supp} \overline{N} \subset \bigcup_{\overline{\langle\alpha\rangle}_{C'} \neq \mathfrak{h}^*} \operatorname{Supp} L(\alpha)_{C'} \subset Z$$

This concludes the proof of the last claim. □

Since M is the union of all N we obtain that $\operatorname{Supp} M \subset Z$. Now let $f \in S\mathfrak{h}$ be zero on Z. Then it follows from (8.13) that $fM = 0$ and hence we are done. □

8.4. **Injective dimension.** In this section we use Theorem 8.3.1 and Corollary 8.2.5 to compute the injective dimension of B^χ. More precisely we will prove the following result.

Theorem 8.4.1. (1) *B^χ satisfies the left (and right) Auslander condition. That is if M is a finitely generated left (right) B^χ-module and if N is a non-zero submodule of $\operatorname{Ext}^j_{B^\chi}(M, B^\chi)$ then $j(N) \geq j$ where $j(N) = \inf_j\{j \mid \operatorname{Ext}^j_{B^\chi}(N, B^\chi) \neq 0\}$.*

(2) *If B^χ is a finitely generated B^χ-module then*

$$j(M) + \operatorname{GKdim} M = \operatorname{GKdim} B^\chi$$

(3) *The left and right injective dimension of B^χ are equal. Furthermore*

$$\begin{aligned}\operatorname{inj\,dim} B^\chi &= \operatorname{GKdim} B^\chi - \min_{\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))} \operatorname{GKdim} L(\alpha)_{B^\chi} \\ &= 2(n - \dim \mathfrak{g}) - \min_{\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))} \dim \overline{\langle\alpha\rangle}_{B^\chi}\end{aligned}$$

Proof. This result follows almost immediately from [13] and [12], once we have Theorem 8.3.1. For simplicity we write B for B^χ. It follows from [18, Thm D] that $\operatorname{gr}_F B$ is Gorenstein. This implies (1) by [13, Rem. 4.5]. (2) follows from [13, Thm 4.4] together with (8.2). To prove (3) we claim first that

$$\operatorname{inj\,dim} B = \max j(M) \tag{8.14}$$

where the maximum runs over all finitely generated B-modules. To show this we have to construct M such that

$$\mu \overset{\text{def}}{=} \operatorname{inj\,dim}{}_B B = j(M)$$

Now by [13, Thm 4.4] we have $\mu = \operatorname{inj\,dim} B_B$. So by definition there exists a finitely generated right B-module N such that $M = \operatorname{Ext}^\mu_B(N, B) \neq 0$. If $j(M) \neq \mu$ then by the Auslander condition $j(M) = \infty$ which is impossible by (2).

Rewriting (8.14) using (2) yields

$$\operatorname{inj\,dim} B = \operatorname{GKdim} B - \min \operatorname{GKdim} M \tag{8.15}$$

Let M be such that $\operatorname{GKdim} M$ is minimal. We may assume that M is simple. Then by Theorem 8.3.1 we have

$$\operatorname{GKdim} M \geq \frac{1}{2} \operatorname{GKdim}(B/\operatorname{Ann} M)$$

Now by Theorem 7.3.1(3) we have $\operatorname{Ann} M = J(\alpha)$ for some $\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ and hence by cor. 8.2.5

$$\frac{1}{2} \operatorname{GKdim}(B/\operatorname{Ann} M) = \operatorname{GKdim} L(\alpha)$$

Hence by the choice of M we find $\operatorname{GKdim} M = \operatorname{GKdim} L(\alpha)$. Substituting this in (8.15) yields the first equality of (3). The second equality in (3) follows from Theorem 8.2.1 and Proposition 8.2.3. $\square$

9. Finite global dimension

9.1. **Introduction and statement of the main result.** In this section the notations will be as in §6,§7. Recall that in §4.4 we introduced the $\to$-relation on $\mathfrak{g}^*$ which was closely related with the Morita equivalences among the various B^χ.

In §7.6 it was shown that minimal χ's correspond precisely to those B^χ's that are simple. Now let us define $\chi \in \mathfrak{g}^*$ to be maximal if $\chi' \to \chi$, $\chi' \in \mathfrak{g}^*$ implies $\chi \to \chi'$ (we think of $\to$ as $\geq$). Our main result in this section will be.

Theorem 9.1.1. *Let $\chi \in \mathfrak{g}^*$. Then B^χ has finite global dimension if and only if χ is maximal.*

Using Theorem 8.4.1(3) and the following easy lemma we may compute the exact value of $\operatorname{gl\,dim} B^\chi$, once we know it is finite.

Lemma 9.1.2. *Assume that C is a Noetherian ring. If $\operatorname{gl\,dim} C$ is finite then $\operatorname{gl\,dim} C = \operatorname{inj\,dim} C$.*

Proof. Clearly $\operatorname{inj\,dim} C \leq \operatorname{gl\,dim} C$ so we have to prove the opposite inequality. Let $\mu = \operatorname{inj\,dim} C$. Assume that M is a finitely generated C-module with

$$0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0$$

a resolution consisting of finitely generated projective C-modules such that $P_l \to P_{l-1}$ is non-split. Suppose $l > \mu$. Dualizing yields that $P_{l-1}^* \to P_l^*$ is surjective and hence split. But then dualizing again yields that $P_l \to P_{l-1}$ must split. This is a contradiction and hence $l \leq \mu$, $\square$

One possibility to prove Theorem 9.1.1 would be to invoke corollary 3.5.11 together with remark 3.5.12. In this way we would have to show that $\operatorname{gl\,dim} H_\Gamma^{(\infty)} < \infty$ for all Γ. Unfortunately we have not been able to do this directly. Instead our arguments are somewhat more complicated.

If χ is *not* maximal, and $\dim \mathfrak{g} = 1$ then it is shown in [28] that $\operatorname{gl\,dim} B^\chi = \infty$ by constructing a module with a periodic projective resolution. This too we have not been able to generalize. Instead we show that $\operatorname{gl\,dim} H_\Gamma^{(\infty)} = \infty$ for some Γ by using the following well-known result.

Proposition 9.1.3. *Let C be a Noetherian ring of finite global dimension and let $G_0(C)$, resp. $K_0(C)$ be the Grothendieck group of finitely generated, resp. finitely generated projective C-modules. Then $G_0(C) = K_0(C)$.*

To prove the other half of Theorem 9.1.1 we use the method of [28]. However, whereas in [28] we could get by with ordinary Öre localization, in this paper we are forced to use the more sophisticated micro-localization. See e.g. [1].

§9.2,§9.3 below are devoted to the proof that maximality implies finite global dimension. §9.4,§9.5 are devoted to the converse.

9.2. **Some useful facts.** In this section we give the tools to construct an analog of the exact sequence in [28, §5]. As was already pointed out in §9.1, we are forced to work with algebraic micro-localization.

The methods in this section are derived from [29]. However we have adapted the notations to make them more compatible with our current situation.

Let P be a Laurent polynomial ring of the form

$$P = k[y_1, \dots, y_d, y_{d+1}^{\pm 1}, \dots, y_{d+e}^{\pm 1}]$$

and let $Y = \operatorname{Spec} P$. Assume that an algebraic torus G acts diagonally on $ky_1 + \cdots + ky_{d+e}$ with weights $(\zeta_i)_{i=1,\dots,d+e} \in X(G) \subset X(G)_{\mathbb{R}}$.

We fix throughout a number $1 \le c \le d$ and we let $\mathcal{W}$ stand for the set $\{1, \dots, c\}$. For $S \subset \mathcal{W}$ we define

$$U_S = \{(y_1, \dots, y_{d+e}) \in Y \mid \forall i \in S : y_i \neq 0\}$$
$$Y_S = \{(y_1, \dots, y_{d+e}) \in Y \mid \forall i \in S : y_i = 0\}$$

Obviously U_S is open and Y_S is closed in Y.

Let

$$G^o = \{g \in G \mid g(y_i) = y_i \text{ for } i = c+1, \dots, d+e\}$$

By ζ_i^o we denote the weight ζ_i restricted to G^o.

If E is a finite dimensional $\mathbb{R}$-vector space and $T \subset E$ then we denote by $\operatorname{pos} T$ the cone spanned by T. That is : the set of all positive linear combinations of elements in T. By $\operatorname{relint} \operatorname{pos} T$ we the denote the relative interior of $\operatorname{pos} T$. This consists of all *strictly* positive linear combinations of elements in T. By convention : $\operatorname{relint} \operatorname{pos} \emptyset = \operatorname{pos} \emptyset = \{0\}$.

Let $\delta \in X(G)_{\mathbb{R}}$ and denote its restriction to G^o by δ^o. We define

$$\mathcal{W}_\delta = \{S \subset \mathcal{W} \mid \delta^o \in \operatorname{relint} \operatorname*{pos}_{i \in S} \zeta_i^o\}$$

It is easy to see that $\mathcal{W}_\delta$ is closed under unions.

For $\psi \in Y(G^o)_{\mathbb{R}}$ we define

$$S_\psi = \{i \in \mathcal{W} \mid \langle \psi, \zeta_i^o \rangle > 0\}$$

Finally we put

$$U_\delta = \bigcup_{S \in \mathcal{W}_\delta} U_S$$
$$Y_\delta = \bigcup_{\substack{\langle \psi, \delta^o \rangle > 0 \\ \psi \in X(G^o)_{\mathbb{R}}}} Y_{S_\psi}$$

Proposition 9.2.1. $Y = U_\delta \sqcup Y_\delta$

Proof. The case $\delta^o = 0$ is trivial so we assume $\delta^o \neq 0$.

First we show that $U_\delta \cap Y_\delta = \emptyset$. To this end it is sufficient to show that for all $S \in \mathcal{W}_\delta$ and for all $\psi \in X(G^o)_{\mathbb{R}}$ such that $\langle \psi, \delta^o \rangle > 0$ one has that $S \cap S_\psi \neq \emptyset$.

Suppose on the contrary that we have found S and S_ψ such that $S \cap S_\psi = \emptyset$. This means that for all $i \in S$ one has $\langle \psi, \zeta_i^o \rangle \leq 0$ and furthermore there exist $(u_i)_i \in \mathbb{R}^+_{>0}$ such that $\delta^o = \sum_{i \in S} u_i \zeta_i^o$. But this clearly in contradicts $\langle \psi, \delta^o \rangle > 0$.

Next we prove that $U_\delta \cup Y_\delta = Y$. Let $(y_1, \ldots, y_{d+e}) \in Y$ and put

$$T_1 = \{i \in \mathcal{W} \mid y_i \neq 0\}$$
$$T_2 = \{i \in \mathcal{W} \mid y_i = 0\}$$

We have to show that one of the following is true.

(1) There exist $S \in \mathcal{W}_\delta$ such that $S \subset T_1$.
(2) There exist $\psi \in X(G^o)_\mathbb{R}$ such that $\langle \psi, \delta^o \rangle > 0$ and $S_\psi \subset T_2$.

Assume that (1) is false. This means that δ^o cannot be written as

$$\delta^o = \sum_{i \in T_1} u_i \zeta_i^o, \qquad u_i \in \mathbb{R}^+$$

or equivalently δ^o does not lie in the cone spanned by $(\zeta_i^o)_{i \in T_1}$.

But then there must exist a "separating hyperplane." That is a $\psi \in X(G^o)_\mathbb{R}$ such that $\langle \psi, \delta^o \rangle > 0$ and $\forall i \in T_1 : \langle \psi, \zeta_i^o \rangle \leq 0$.

Now let $i \in S_\psi$. Then $\langle \psi, \zeta_i^o \rangle > 0$ and hence $i \notin T_1$. Therefore $i \in T_2$, whence (2) is true. $\square$

The augmented Čech complex

$$\mathcal{C}_\delta = \mathcal{C}^\cdot(\mathcal{O}_Y; (U_S)_{S \in \mathcal{W}_\delta})$$

is given by

$$\mathcal{C}_\delta(P)_q = \bigoplus_{\{S_{i_1}, \ldots, S_{i_q}\} \subset \mathcal{W}_\delta} \Gamma(U_{S_{i_1}} \cap \cdots \cap U_{S_{i_q}}, \mathcal{O}_Y)$$

with the usual alternating boundary maps.

By Proposition 9.2.1 the homology of $\mathcal{C}_\delta(P)$ is given by $H^*_{Y_\delta}(Y, \mathcal{O}_Y)$.

Now using the techniques of [29, §3.4] (see also [27, lemma 6.6]) one sees that $H^*_{Y_\delta}(Y, \mathcal{O}_Y)$ may be suitably filtered such that the associated quotients have the form $H^*_{Y_{S_\psi}}(Y, \mathcal{O}_Y)$, $\psi \in X(G^o)_\mathbb{R}$, $\langle \psi, \delta^o \rangle > 0$.

Furthermore by an obvious generalization of [29, Cor, 3.3.2] the G^o-weights of $H^*_{Y_{S_\psi}}(Y, \mathcal{O}_Y)$ are given by

$$\sum_{i=1}^{c} a_i \zeta_i^0, \qquad (a_i)_i \in \mathbb{Z} \tag{9.1}$$

where

$$a_i < 0 \quad \text{if} \quad i \in S_\psi$$
$$a_i \geq 0 \quad \text{if} \quad i \in \{1, \ldots, c\} \setminus S_\psi$$

Now if

$$\sum_{i=1}^{c} \mathbb{R}^+ \zeta_i + \sum_{i=c+1}^{d+e} \mathbb{R} \zeta_i = X(G)_\mathbb{R} \tag{9.2}$$

then $\sum_{i=1}^{c} \mathbb{R}^+ \zeta_i^o = X(G^o)_\mathbb{R}$ and hence S_ψ is never empty. Applying $\langle \psi, - \rangle$ gives that (9.1) can never yield zero. So we have shown that $H^*_{Y_{S_\psi}}(Y, \mathcal{O}_Y)^G = 0$ for all $\psi \in X(G^o)_\mathbb{R}$, $\langle \psi, \delta^o \rangle > 0$. Hence the foregoing discussion yields.

Theorem 9.2.2. *Assume* (9.2). *Then* $\mathcal{C}_\delta(P)^G$ *is exact.*

Now let us briefly recall the notion of algebraic micro-localization. As general references we use [32] and [1].

Let $(F_nA)_{n\in\mathbb{Z}}$ be an ascending filtration on a ring A and let $S \subset \operatorname{gr}_F A$ be an Öre set consisting of homogeneous elements. Then the micro-localization $Q^\mu_S(A)$ is a filtered ring together with a filtered structure morphism $A \to Q^\mu_S(A)$ having the properties.

(1) If $s \in F_mA \setminus F_{m-1}A$ such that $\bar{s} \in S$ then s is invertible in $Q^\mu_S(A)$.
(2) $Q^\mu_S(A)$ is complete.
(3) $Q^\mu_S(A)$ is universal with respect to properties (1) and (2).

It is shown in [1] and [32] that $Q^\mu_S(A)$ exists and is unique. Further remarkable properties of $Q^\mu_S(A)$ are collected in loc. cit. Let us mention that $\operatorname{gr}_F Q^\mu_S(A) = (\operatorname{gr}_F A)_S$ and if $\tilde{A} = \bigoplus_n F_nA$ is Noetherian then $Q^\mu_S(A)$ is a flat A-module. Note that if $F_nA = 0$ for $n < 0$ then $\tilde{A}$ is Noetherian if and only if $\operatorname{gr}_F A$ is Noetherian.

If S is obtained from some Öre set T in A then $Q^\mu_S(A)$ is equal to the completion of A_T. If $\tilde{A}$ is Noetherian then we may always take for T the largest possible multiplicative set mapping to S. This is the so-called saturation of S, denoted by S^{sat}. It is shown in [1] that S^{sat} is an Öre set in A.

In the sequel we will need graded analogs of the above notions. So we assume that A is graded by some as yet unspecified group and that all F_nA are also graded. Furthermore we assume that S consists of homogeneous elements for both gradings on $\operatorname{gr}_F A$. Then one may construct a graded micro-localization $Q^{\mu,\text{gr}}_S(A)$ satisfying the graded analogs of (1)(2)(3) and having properties analogous to that of ungraded micro-localization.

We will use graded algebraic micro-localization in the following situation. A will be filtered such that $\operatorname{gr}_F A = P$ where P is as above. We assume in addition that A is graded by the character group of some algebraic torus T acting rationally on A such that all F_nA are graded and such that the $y_i \in P$ are homogeneous. Finally we assume that G acts on A through an inclusion $G \subset T$.

For $S \subset \mathcal{W} = \{1,\dots,c\}$ we let $A_{\mu,S}$ stand for the graded algebraic micro-localization of A at the multiplicative set generated by $\{y_i \mid i \in S\}$. Then $\operatorname{gr}_F A_{\mu,S}$ is the localization of P at $\{y_i \mid i \in S\}$. That is $\operatorname{gr}_F A_{\mu,S} = \Gamma(U_S, \mathcal{O}_Y)$.

We now define the augmented "micro Čech complex" $\mathcal{C}_{\mu,\delta}(A)$ by

$$\mathcal{C}_{\mu,\delta}(A)_q = \bigoplus_{\{S_{i_1},\dots,S_{i_q}\}\subset\mathcal{W}_\delta} A_{\mu,S_{i_1}\cup\cdots\cup S_{i_q}}$$

with the usual alternating boundary maps.

Proposition 9.2.3. *If* (9.2) *is satisfied then* $\mathcal{C}_{\mu,\delta}(A)^G$ *is exact.*

Proof. We have

$$\operatorname{gr} \mathcal{C}_{\mu,\delta}(A)^G = \mathcal{C}_\delta(P)^G$$

which is exact by Theorem 9.2.

Now $\mathcal{C}_{\mu,\delta}(A)^G$ has (graded) complete terms. This implies in the usual way that $\mathcal{C}_{\mu,\delta}(A)^G$ is also exact. □

9.3. **Rings of differential operators of finite global dimension.** We now use the notations of §6,§7. As before we consider the weights $\eta_1, \dots, \eta_n \in X(G)$ implicitly also as elements of $X(G)_{\mathbb{Q}}$, $X(G)_{\mathbb{R}}$, $\mathfrak{g}^*$, ...

To continue it will be convenient to introduce alternative names for $x_1, \dots, x_n$, $\partial_1, \dots, \partial_n$. We put

$$y_1 = \partial_1, \dots, y_r = \partial_r$$
$$y_{r+1} = x_1, \dots, y_{2r} = x_r$$
$$y_{2r+1} = \partial_{r+1}, \dots, y_{2r+s} = \partial_{r+s}$$
$$y_{2r+s+1} = x_{r+1}, \dots, y_{2r+2s} = x_{s+r}$$

To make our notations compatible with §9.2 we put $d = 2r + s$, $e = s$. For c we take $2r$. We filter A, $A^{\mathfrak{g}}$, B^{χ}, etc... with the filtration F which was introduced in §8.2.

For $\chi \in \mathfrak{g}^*$, $S \subset \mathcal{W}$ we put

$$A^{\mathfrak{g}}_{\mu,S} = \oplus_{\alpha \in V(\mathfrak{g})} A_{\mu,S,\alpha} \tag{9.3}$$

and

$$B^{\chi}_{\mu,S} = A^{\mathfrak{g}}_{\mu,S}/(\mathfrak{g} - \chi(\mathfrak{g}))A^{\mathfrak{g}}_{\mu,S} \tag{9.4}$$

(compare with §4.4).

We will call $S \subset \mathcal{W}$ reduced if it contains no pair of the form $\{i, i + r\}$. In that case $\{y_i \mid i \in S\}$ generates an Öre set and we will use the notations in (9.3)(9.4) also without the μ-symbol, thereby refering to ordinary Öre localization.

Lemma 9.3.1. $(\mathfrak{g} - \chi(\mathfrak{g}))$ *is generated by a regular sequence in* $A^{\mathfrak{g}}_{\mu,S}$.

Proof. Clearly $(\mathfrak{g} - \chi(\mathfrak{g}))$ is generated by a regular sequence in D. Since A is a direct sum of copies of D it is D-flat. Furthermore since $\operatorname{gr}_F A$ is Noetherian, $A_{\mu,S}$ is A-flat and so $A_{\mu,S}$ is also D-flat. So $(\mathfrak{g} - \chi(\mathfrak{g}))$ is also generated by an $A_{\mu,S}$-regular sequence. Observing that $A^{\mathfrak{g}}_{\mu,S}$ is a D-direct summand of $A_{\mu,S}$ concludes the proof. □

Now for $\delta \in X(G)_{\mathbb{R}}$ we define

$$\mathcal{C}_{\mu,\delta}(B^{\chi}) = \mathcal{C}_{\mu,\delta}(A)^G/(\mathfrak{g} - \chi(\mathfrak{g}))\mathcal{C}_{\mu,\delta}(A)^G$$

where $\mathcal{C}_{\mu,\delta}(A)^G$ is defined in §9.2.

Proposition 9.3.2. $\mathcal{C}_{\mu,\delta}(B^{\chi})$ *is exact.*

Proof. By the fact that the $(\zeta_i)_{1,\dots,2r}$ come in pairs $\pm\eta_i$, (9.2) is implied by the fact that $\sum k\eta_i = \mathfrak{g}^*$. Hence by Proposition 9.2.3, $\mathcal{C}_{\mu,\delta}(A)^G$ is exact.

Lemma 9.3.1 implies that $(\mathfrak{g} - \chi(\mathfrak{g}))$ is generated by a regular sequence on the terms of the complex $\mathcal{C}_{\mu,\delta}(A)^G$. One then shows by induction that $\mathcal{C}_{\mu,\delta}(B^{\chi})$ is also exact. □

To be able to continue we must understand better $B^{\chi}_{\mu,S}$. As a first step, but also as a useful example we compute $Q^{\mu,\mathrm{gr}}_S(A)$ where $A = k[x, \partial]$ graded by $\deg x = -\deg \partial = 1$ and $S = \{1\}, \{x\}, \{\partial\}, \{x, \partial\}$.

1, x and ∂ generate Öre sets in A and the filtrations on the homogeneous parts of A, A_x, A_∂ are left limited. In particular the homogeneous parts of A, A_x, A_∂ are complete and therefore $Q^{\mu,\mathrm{gr}}_S(A)$ is simply equal to A_S if $S = \{1\}, \{x\}, \{\partial\}$.

The computation of $Q^{\mu,\mathrm{gr}}_{x\partial}(A)$ is more interesting. Let $\hat{D}_\infty = k((\pi^{-1}))$ where as usual $D = k[\pi]$, $\pi = x\partial$. Then one verifies that the multiplication on A extends uniquely to a continuous multiplication on $\hat{D}_\infty \otimes_D A$ and

$$Q^{\mu,\mathrm{gr}}_{x\partial}(A) = \hat{D}_\infty \otimes_D A$$

Of course one also has

$$Q^{\mu,\mathrm{gr}}_{x\partial}(A) = \hat{D}_\infty \otimes_D A_x$$
$$Q^{\mu,\mathrm{gr}}_{x\partial}(A) = \hat{D}_\infty \otimes_D A_\partial$$

These examples serve to illustrate that in order to compute $A_{\mu,S}$ for $S \subset \{1, \dots, 2r\}$, we may always reduce to the case that S is reduced. One proves the following result.

Proposition 9.3.3. (1) *Let $S_0 \subset S$ be obtained from $S \subset \{1, \dots, 2r\}$ by removing all pairs $\{i, i+r\}$ and let $S_0 \subset S_1 \subset S$ be reduced. Then*

$$B^\chi_{\mu,S} = \left(\widehat{\bigotimes_{\{i,i+r\}\subset S}} \hat{D}_{i,\infty} \right) \otimes_D B^\chi_{S_1} \tag{9.5}$$

where $\hat{\otimes}$ denotes the completed tensor product and $\hat{D}_{i,\infty} = k((\pi_i^{-1}))$.

(2) *If S is reduced then as a right B^χ-module*

$$B^\chi_S = \varinjlim_t B^{\chi',\chi}$$

where $\chi' = \chi + t\delta$, $t \in \mathbb{N}$ and δ is some strictly positive integer linear combination of $(\zeta_i)_{i\in S}$.

Proof. (1) The discussion above yields that (9.5) holds with $B^\chi_{\mu,S}$ replaced by $A_{\mu,S}$. Taking G-invariants and quotienting out by $(\mathfrak{g} - \chi(\mathfrak{g}))$ yields the desired result.

(2) Assume that $\delta = \sum_{i\in S} \delta_i \zeta_i$, $\delta_i > 0$. Let $s = \prod_{i\in S} y_i^{\delta_i}$. The powers of s form an Öre set and hence we can localize at s.

Then

$$A_S = A_s = \varinjlim_t s^{-t} A$$

and hence using the notation of §4.4

$$A^{\mathfrak{g}}_S = \varinjlim_t s^{-t} A^{\mathfrak{g}}_{t\delta} \cong \varinjlim_t A^{\mathfrak{g}}_{t\delta}$$

as right $A^{\mathfrak{g}}$-modules. Tensoring on the right with B^χ yields

$$\begin{aligned} B^\chi_S &= \varinjlim_t A^{\mathfrak{g}}_{t\delta} / A^{\mathfrak{g}}_{t\delta}(\mathfrak{g} - \chi(\mathfrak{g})) \\ &= \varinjlim_t A^{\mathfrak{g}}_{t\delta} / (\mathfrak{g} - (\chi + t\delta)(\mathfrak{g})) A^{\mathfrak{g}}_{t\delta} \\ &= \varinjlim_t B^{\chi+t\delta,\chi} \quad \square \end{aligned}$$

Proposition 9.3.4. *Assume that $\delta \in \sum \mathbb{Z}\eta_i$ and $\chi \in \mathfrak{g}^*$ are such that for all $n \geq 0$ one has $\chi \to \chi + n\delta$. Then if $S \in \mathcal{W}_\delta$, $B^\chi_{\mu,S}$ is a right flat B^χ-module.*

Proof. Since $S \in \mathcal{W}_\delta$ there exists $m \in \mathbb{N} \setminus \{0\}$ such that

$$m\delta = \sum_{i \in S} a_i \zeta_i + \sum_{i \in \{c+1,\dots,d+e\}} b_i \zeta_i$$

with $(a_i)_{i \in S} \in \mathbb{N} \setminus \{0\}$, $(b_i)_i \in \mathbb{Z}$. If one replaces δ by $m\delta - \sum_{i \in \{c+1,\dots,d+e\}} b_i \zeta_i$ then using Theorem 4.4.4 one deduces that one still has for $n \geq 0 : \chi \to \chi + n\delta$ but now δ is a positive integer linear combination of $(\zeta_i)_{i \in S}$. Furthermore by using $\zeta_i = -\zeta_{i+r}$ we can assume that the coefficients of ζ_i or ζ_{i+r} or both are zero. Then if we put $S_1 = \{i \in S \mid a_i \neq 0\}$ we have that S_1 is reduced.

By Proposition 9.3.3(1) we have that as right $B^\chi_{S_1}$-module

$$B^\chi_{\mu,S} = \left(\widehat{\bigotimes_{\{i,i+r\} \subset S}} \hat{D}_{i,\infty} \right) \otimes_D B^\chi_{S_1}$$

Since $\widehat{\bigotimes}_{\{i,i+r\} \subset S} \hat{D}_{i,\infty}$ is easily seen to be a flat D-module, it suffices to show that $B^\chi_{S_1}$ is a flat B^χ-module.

By hypothesis $\forall n \geq 0$, $\chi \to \chi' = \chi + n\delta$. General yoga [15, 3.5.4] about the Morita context 4.5 yields that $B^{\chi',\chi}$ is right projective. By Proposition 9.3.3(2) we obtain that $B^\chi_{S_1}$ is right flat. □

Proposition 9.3.5. *Assume that $S \subset \mathcal{W}$ is reduced and has the property that*

$$\sum_{i \in S} k\zeta_i + \sum_{i \in \{c+1,\dots,d+e\}} k\zeta_i = \mathfrak{g}^*$$

Then B^χ_S has finite global dimension for all $\chi \in \mathfrak{g}^*$.

Proof. Using the automorphisms $x_i \to \partial_i$, $\partial_i \to -x_i$ we may assume that $S \subset \{r+1,\dots,2r\}$. By hypothesis $(\zeta_i)_{i \in S \cup \{c+1,\dots,d+e\}}$ contains a basis for $\mathfrak{g}^*$. Now after possibly renumbering variables and taking different r, s we can apply lemma 9.3.6 below to conclude that B^χ_S is of the form W^H where H is a finite group and W is a Weyl algebra with some of the variables inverted.

Furthermore one verifies that H acts faithfully on $\operatorname{gr}_F W$ and hence H acts by outer automorphisms on W. Therefore by [15, 7.8.11, 7.8.12] W^H has finite global dimension. □

Lemma 9.3.6. *Assume that $\eta_{t+1}, \cdots, \eta_n$ forms a basis for $\mathfrak{g}^*$ for some $t \geq r$. Put*

$$W = k[x_1,\dots,x_r,x_{r+1}^{\pm 1},\cdots x_t^{\pm 1},\partial_1,\dots,\partial_t]$$

W is $\mathbb{Z}^t$-graded in the standard way. Define

$$C = \bigoplus_{\sum_{i=1}^t \alpha_i \eta_i \in \sum_{i=t+1}^n \mathbb{Z}\eta_i} W_\alpha$$

If $C_\alpha \neq 0$ define $(v_i)_{i=t+1,\dots,r}$ by $\sum_{i=1}^t \alpha_i \eta_i = \sum_{i=t+1}^n v_i \eta_i$. Then the map

$$C \to B^\chi : r \mapsto x_{t+1}^{-v_{t+1}} \cdots x_n^{-v_n} r$$

is an an isomorphism of k-algebras.

The proof of this lemma is left to the reader.

Lemma 9.3.7. *Let $\chi \in \mathfrak{g}^*$. Then there exist $\delta_1,\dots,\delta_t \in \sum_{i=1}^n \mathbb{Z}\eta_i$ that form a basis for $\mathfrak{g}^*$ such that for any $\delta = \sum_i u_i \delta_i$, $u_i \in \mathbb{N}$ one has $\chi + \delta \to \chi$.*

Proof. For any $\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ we let F_α be the semi-group of all $\sum_{i=1}^n v_i\eta_i$, $(v_i)_i \in \mathbb{Z}$ where

$$\forall i \in \{1,\dots,r\}, \alpha_i \in \mathbb{Z} : \begin{array}{l} \alpha_i \geq 0 \Rightarrow v_i \geq 0 \\ \alpha_i < 0 \Rightarrow v_i \leq 0 \end{array} \tag{9.6}$$

Let $\delta \in \bigcap_\alpha F_\alpha$ then for all $\alpha \in V(\mathfrak{g})$ one may write $\delta = \sum_{i=1}^n v_i\eta_i$, $v_i \in \mathbb{Z}$ satisfying (9.6).

Put $\beta = \alpha + v$. Then $\alpha \iff \beta$ and $\beta \in V(\mathfrak{g} - (\chi+\delta)(\mathfrak{g}))$. According to Theorem 4.4.4 we then have $\chi + \delta \to \chi$. So clearly we have to show that $\bigcap_\alpha F_\alpha$ contains a basis for $\mathfrak{g}^*$. Note in passing that there are only a finite number of different F_α's.

We will now construct elements of $\bigcap_\alpha F_\alpha$. We will use again compatible projections $\mathrm{pr} : k \to \mathbb{Q}$, $\mathrm{pr} : \mathfrak{g}^* \to \mathbb{Q}^t$. $\mathrm{pr} : \mathfrak{t} \to \mathbb{Q}^n$ as in §7.5.

Let $\epsilon \in (\mathbb{Q}^+)^n$ have the property that $\forall i : 0 \leq \epsilon_i < 1$ and set $\mu = \chi + \sum_{i=1}^n \epsilon_i\eta_i$. Note that if $\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ then $\mu = \sum_i \beta_i\eta_i$ where $\beta_i = \alpha_i + \epsilon_i$ and α_i have the same sign.

Clearly there exists some $m_\alpha \in \mathbb{N}\backslash\{0\}$ such that $m_\alpha\mu \in F_\alpha$. Let $m = \mathrm{lcm}_{F_\alpha} m_\alpha$. Then $m\mu \in \bigcap_\alpha F_\alpha$.

Since $m\mu = m\,\mathrm{pr}(\chi) + \sum_{i=1}^n m\epsilon_i\eta_i$ it is clear that by varying ϵ_i we may obtain a basis for $\mathfrak{g}^*$. $\square$

Corollary 9.3.8. *Let $\chi \in \mathfrak{g}^*$. Then there exist $\delta \in \sum_{i=1}^n \mathbb{Z}\eta_i$ such that for all $n \in \mathbb{N}$ one has $\chi + n\delta \to \chi$ and if $\delta = \sum_{i=1}^n v_i\eta_i$, $v_i \in \mathbb{R}$ then $(\eta_i)_{v_i\neq 0}$ spans $\mathfrak{g}^*$.*

Proof. The set of all δ that may be written as $\sum_{i=1}^n v_i\eta_i$ such that $(\eta_i)_{v_i\neq 0}$ does *not* span $\mathfrak{g}^*$ is contained in a finite number of subspaces of $\mathfrak{g}^*$ and hence is not Zariski dense.

On the other hand we know by lemma 9.3.7 that the δ that have the property that $\chi + n\delta \to \chi$, $n \geq 0$ *are* Zariski dense. Hence there must be a δ satisfying the requirements of the corollary. $\square$

Theorem 9.3.9. *Assume that $\chi \in \mathfrak{g}^*$ is maximal. Then B^χ has finite global dimension.*

Proof. We choose $\delta \in \sum \mathbb{Z}\eta_i$ as in corollary 9.3.8. Since χ is maximal we have $\chi \to \chi + n\delta$ for all $n \geq 0$.

Then by Proposition 9.3.2 and 9.3.4 $\mathcal{C}_{\mu,\delta}(B^\chi)$ is an exact complex of the form $0 \to B^\chi \to \cdots$, consisting of right flat B^χ-modules.

Now let $S \in \mathcal{W}_\delta$. If S is reduced then by Proposition 9.3.5, $B^\chi_{\mu,S} = B^\chi_S$ has finite global dimension. If this were true for all $S \in \mathcal{W}_\delta$ then we could finish the proof as in [28, §5].

Unfortunately we don't know how to do this, and therefore we have to make a slight detour.

For each $S \in \mathcal{W}_\delta$ let S_1 be a set with the property that for any pair $\{i, i+r\} \subset S$ one has $|S_1 \cap \{i, i+r\}| = 1$.

Since $S \in \mathcal{W}_\delta$ we have by definition that

$$\sum_{i\in S} k\zeta_i + \sum_{i\in\{c+1,\dots,d+e\}} k\zeta_i = \mathfrak{g}^*$$

Clearly this condition is still true if we replace S by S_1. So by Proposition 9.3.5, $B^\chi_{S_1}$ has finite global dimension. Furthermore by Proposition 9.3.3(1), $B^\chi_{\mu,S}$ is a right flat $B^\chi_{S_1}$-module.

Now we will modify the proof in [28, §5] as follows. Let $M \in B^\chi$-mod be finitely generated. We will show that M has finite injective dimension.

Since $\mathcal{C}_{\mu,\delta}(B^\chi) \otimes_{B^\chi} M$ is exact it suffices to show that each $B^\chi_{\mu,S} \otimes_{B^\chi} M$, $S \in \mathcal{W}_\delta$ has finite injective dimension. Now $B^\chi_{\mu,S} \otimes_{B^\chi} M = B^\chi_{\mu,S} \otimes_{B^\chi_{S_1}} B^\chi_{S_1} \otimes_{B^\chi} M$ and hence by replacing $B^\chi_{S_1} \otimes_{B^\chi} M$ by a finite resolution, consisting of finitely generated projective $B^\chi_{S_1}$-modules it suffices to show that $B^\chi_{\mu,S}$ has finite injective dimension as B^χ-module (here one uses of course that any finitely generated projective module is a direct summand of a free module of finite rank). Now since $B^\chi_{\mu,S}$ is a right flat B^χ-module, every injective $B^\chi_{\mu,S}$-module is injective as B^χ-module. Hence it is sufficient to show that $B^\chi_{\mu,S}$ has finite injective dimension over itself.

Now $B^\chi_{\mu,S}$ is (graded) complete and hence the spectral sequence

$$\operatorname{Ext}^*_{\operatorname{gr}_F B^\chi_{\mu,S}}(\operatorname{gr}_F M, \operatorname{gr}_F B^\chi_{\mu,S}) \Rightarrow \operatorname{Ext}^*_{B^\chi_{\mu,S}}(M, B^\chi_{\mu,S})$$

for a finitely generated $B^\chi_{\mu,S}$-module M, equipped with a good filtration F, yields that it is sufficient to show that $\operatorname{gr}_F B^\chi_{\mu,S}$ has finite injective dimension.

Now

$$\operatorname{gr}_F B^\chi_{\mu,S} = P_S^G/(\mathfrak{g})P_S^G$$

where P_S is the localization of P at $(y_i)_{i\in S}$.

Since $\mathfrak{g}$ is generated by a regular sequence in P_S, it suffices to show that P_S^G is Gorenstein. This follows from [18, Thm 4.6]. □

9.4. **On some orders of infinite global dimension.** In the next two sections we will complete the proof of Theorem 9.1.1 by proving the converse to 9.3.9. We start by giving some results on certain orders over complete regular local rings that might be of independent interest.

Let $R = k[[\pi]]$ and

$$H = \begin{pmatrix} R & (\pi) \\ R & R \end{pmatrix}$$

H is the completed path algebra of the quiver

$$\begin{tikzcd} -1 \bullet \arrow[r, shift left] & \bullet\ 0 \arrow[l, shift left] \end{tikzcd}$$

where -1 and 0 serve as labels.

Let $p \in \mathbb{N}$. With H^p we denote the p-fold completed tensor product $H^{\hat{\otimes} p}$. Let $Q^p = \{-1, 0\}^p$. We make Q^p into a quiver by introducing an arrow $v \to w$ from $v = (v_1, \dots, w_1)$ to $w = (w_1, \dots, w_r)$ if there is exactly one i such that $v_i \neq w_i$. We also introduce the relations $u \to v \to w = u \to v' \to w$ if u and w differ in exactly two places and $v \neq v'$ is such that the indicated arrows are defined. (Note in passing that we still write a path $\xrightarrow{a}\xrightarrow{b}$ as ba.) Then H^p is the completed path algebra of Q^p.

We will call a path

$$u_1 \to u_2 \to \cdots \to u_k$$

reduced if for every i, the i'th coordinate $(u_j)_i$ changes at most once on the path. Clearly every reduced path from u_1 to u_k is equivalent modulo the relations and hence gives rise to the same element of H^p.

If $v \in Q^p$ then we denote by e_v the corresponding idempotent in H^p. Similarly if $S \subset Q^p$ then $e_S = \sum_{v\in S} e_v$.

Below we will use the following result

Lemma 9.4.1. *Let $v, w \in Q^p$. Then*

$$H^p e_v H^p e_w H^p = H^p x H^p$$

where x is represented by a reduced path from w to v.

Proof. Left to the reader. □

If $K \subset \{1, \dots, p\}$ then there is a projection map

$$\mathrm{pr}_K : Q^p \to Q^{|K|} : (v_1, \dots, v_p) \mapsto (v_i)_{i \in K}$$

We will call the fibers of these maps the faces of Q^p.

Lemma 9.4.2. *Let $S \subset Q^p$. Then $G_0(H^p/H^p e_S H^p)$ is* rationally *generated by the classes of*

$$H^p/H^p e_T H^p \tag{9.7}$$

where $S \subset T$ and $Q^p \setminus T$ is a face.

Proof. We use reverse induction on $|S|$. So the initial step is $S = Q^p$ and hence $e_S = 1$. In this case there is nothing to prove.

Now we consider the case that $Q^p - S$ is a face. Then

$$H^p/H^p e_S H^p = H^q$$

for $2^q = |Q^p - S|$ and hence we may assume without loss of generality that $S = \emptyset$.

Since H^p has finite global dimension we have

$$G_0(H^p) = K_0(H^p) = K_0(H)^{\otimes p} = G_0(H)^{\otimes p} \tag{9.8}$$

Inspection reveals that the isomorphism

$$G_0(H)^{\otimes p} \to G_0(H^p)$$

obtained from (9.8) is given by

$$[M_1] \otimes \cdots \otimes [M_p] \to [M_1 \mathbin{\hat{\otimes}_k} M_2 \mathbin{\hat{\otimes}_k} \cdots \mathbin{\hat{\otimes}_k} M_p] \tag{9.9}$$

Now let $P_v = He_v$, $S_v = P_v/\operatorname{rad}(P_v)$, $v \in \{-1, 0\}$. Then we have a projective resolution

$$0 \to P_0 \to P_{-1} \to S_{-1} \to 0$$

and furthermore $H = P_0 \oplus P_{-1}$.

Therefore in $G_0(H) \otimes \mathbb{Q}$.

$$[P_{-1}] = \frac{1}{2}([H] + [S_{-1}])$$
$$[P_0] = \frac{1}{2}([H] - [S_{-1}])$$

Hence rationally $G_0(H)$ is generated by $[H], [S_{-1}]$.

Now using the fact that $H = H/He_\emptyset H$, $S_{-1} = H/He_0 H$ we obtain that completed tensor products of these modules are of the form (9.7) and then (9.9) implies that such completed tensor products rationally generate $G_0(H^p)$. This finishes the case $S = \emptyset$.

Now we consider the case where $Q^p - S$ is not a face. Then there must exist $v, w \in Q^p - S$ and a reduced path from w to v such that x meets S (exercise !).

Claim . $H^p e_v H^p e_w H^p \subset H^p e_S H^p$

Proof. By lemma 9.4.1

$$H^p e_v H^p e_w H^p = H^p x H^p$$

This proves the claim since $H^p x H^p \subset H^p e_S H^p$. □

Put

$$A = H^p / H^p e_S H^p$$
$$I = \overline{H^p e_{S\cup\{v\}} H^p} \subset A$$
$$J = \overline{H^p e_{S\cup\{w\}} H^p} \subset A$$

Then the claim implies that $IJ = 0$.

If $M \in A$-mod then there is an exact sequence

$$0 \to JM \to M \to M/JM \to 0$$

which implies that

$$G_0(A/I) \oplus G_0(A/J) \to G_0(A) \tag{9.10}$$

is surjective.

By induction we may assume that $G_0(A/I)$ and $G_0(A/J)$ are rationally generated by classes of the form (9.7). Then by surjectivity of (9.10) we may assume that the same is true for $G_0(A)$. □

Two fibers of the same pr_K are said to be parallel. If $U \subset Q^p$ then we denote by $F(U)$ the number of parallelism classes of faces contained in U. That is we count faces in U, counting parallel faces only once.

Concerning the behavior or $F(U)$ we have the following conjecture.

Conjecture 9.4.3. $F(U) \le |U|$ with equality iff for all $K \subset \{1 \dots p\}$, $q = |K|$ the set $Q^q - \mathrm{pr}_K(Q^p - U)$ is connected.

Some of the arguments below would simplify if this conjecture were true.

Now we prove the following results.

Proposition 9.4.4. *Let $S \subset Q^p$. Then*

$$\mathrm{rk}_{\mathbb{Z}}(G_0(e_S H^p e_S)) = |Q^p| - F(Q^p - S)$$

Proof. We use the exact sequence

$$G_0(H^p / H^p e_S H^p) \to G_0(H^p) \to G_0(e_S H^p e_S) \to 0$$

By lemma 9.4.2 it suffices to show that the classes of the form (9.7) generate a subgroup of rank $F(Q^p - S)$ in $G_0(H^p)$. We use the isomorphism (defined in the proof of lemma 9.4.2)

$$G_0(H) \otimes \cdots \otimes G_0(H) \to G_0(H^p)$$

given by the completed tensor product.

To resolve some ambiguity of notation we will denote the product of $[M_1], \dots, [M_p] \in G_0(H)$ in $G_0(H^p)$ by $[M_1]^{(1)} \cdots [M_p]^{(p)}$. We also put $[H] = 1$.

Let $Q^p - T$ be a face defined by $\mathrm{pr}_K^{-1}(v)$, $K \subset \{1, \dots, p\}$, $v \in Q^{|K|}$. Then

$$[M_T] \overset{\text{def}}{=} [H^p / H^p e_T H^p] = \prod_{i \in K} [H / H e_{-1-v_i} H]^{(i)}$$

Below let S_v, P_v be as in the proof of lemma 9.4.2. Then

$$H / H e_{-1-v_i} H = S_{v_i}$$

so that we obtain

$$[M_T] = \prod_{i\in K} [S_{v_i}]^{(i)}$$

Now $[S_0] = -[S_1]$ in $G_0(H)$ and hence parallel faces yield, up to sign, the same element of $G_0(H^p)$.

Now for $K \subset \{1,\dots,p\}$ let

$$T_K = \mathrm{pr}_K^{-1}(-1,\dots,-1)$$

The proof of the proposition is finished if we can show that all $[M_{T_K}]$ are independent in $G_0(H^p)$. Using the fact that $[S_{-1}] = [P_{-1}] - [P_0] = 2[P_{-1}] - 1$ we obtain

$$[M_{T_K}] = \prod_{i\in K} (2[P_{-1}]^{(i)} - 1) \tag{9.11}$$

So $[M_{T_K}]$ is a linear combination of terms of the form

$$\prod_{i\in L} [P_{-1}]^{(i)} \tag{9.12}$$

with $L \subset K$ and with "longest" term equal to $2^{|K|}\prod_{i\in K}[P_0]^{(i)}$.

Hence if we can show that the elements of the form (9.12) with $L \subset \{1,\dots,p\}$ are independent in $G_0(H^p)$ then we are done.

There are $2^p = \operatorname{rk} G_0(H^p)$ such elements, so it is sufficient to show that they generate $G_0(H^p)$. Now $G_0(H^p) = K_0(H^p)$ has a basis consisting of elements $\prod_{i=1}^r [P_{v_i}]^{(i)}$ for $v = (v_1,\dots,v_r) \in Q^{(p)}$.

Using the relations it is clear that one can express these basis elements in terms of the elements (9.12). $\square$

Corollary 9.4.5. *If $e_S H^p e_S$ has finite global dimension then*

$$F(Q^p - S) = |Q^p - S|$$

Proof. If $e_S H^p e_S$ has finite global dimension then $K_0(e_S H^p E_S) = G_0(e_S H^p e_S)$. Now $K_0(e_S H^p e_S) = \mathbb{Z}^{|S|}$ and by Proposition 9.4.4

$$\operatorname{rk}_{\mathbb{Z}} G_0(e_S H^p e_S) = |Q^p| - F(Q^p - S)$$

This shows what we want. $\square$

9.5. **Rings of differential operators of infinite global dimension.** We now revert to the notations of §6,§7. Let $\theta \in V(\mathfrak{g} - \chi(\mathfrak{g}))$, $\Lambda = \theta + \operatorname{Supp} A$, $\Gamma = \theta + \operatorname{Supp} B^\chi$. Clearly $\Gamma = \Lambda \cap V(\mathfrak{g} - \chi(\mathfrak{g}))$.

In order to apply corollary 3.5.11 we have to understand $H_\Gamma^{(\infty)}$. The answer is given by Proposition 4.4.1

$$H_\Gamma^{(\infty)} = e_{\Lambda,\Gamma} H_\Lambda^{(\infty)} e_{\Lambda,\Gamma}/(\psi(\mathfrak{g})) \tag{9.13}$$

$H_\Lambda^{(\infty)}$ itself was computed in §6. We find that

$$H_\Lambda^{(\infty)} = H^p \mathbin{\hat\otimes} k[[(\pi_i)_{i\notin T}] \tag{9.14}$$

where H^p was introduced in §9.4,

$$T = \{1,\dots,r\} \cap \{i \mid \theta_i \in \mathbb{Z}\}$$

and $p = |T|$.

As was explained in §9.4, H^p is the completed path algebra of the quiver Q^p. We index the vertices of Q^p by elements of $\{-1,0\}^T$.

By Proposition 4.3.1(3), $e_{\Lambda,\Gamma} = \sum_{v\in S_\Gamma} e_v$ where S_Γ is the set of all $v \in \{-1,0\}^T$ such that there exists $\alpha \in \Gamma$ with

$$\forall i \in T: \begin{array}{l} v_i = 0 \Rightarrow \alpha_i \geq 0 \\ v_i = -1 \Rightarrow \alpha_i < 0 \end{array}$$

Theorem 9.5.1. *If B^χ has finite global dimension then*

$$F(Q^p - S_\Gamma) = |Q^p - S_\Gamma| \tag{9.15}$$

for all $\theta \in V(\mathfrak{g} - \chi(\mathfrak{g}))$.

Proof. If B^χ has finite global dimension then by corollary 3.5.11 and remark 3.5.12, $H_\Gamma^{(\infty)}$ has finite global dimension for all Γ.

Now $H_\Gamma^{(\infty)}$ is the completion of H_Λ at the ideal $\psi(m_0) \subset \psi(D)$ (§3.5). Since $\mathfrak{g} \subset m_0$ this implies that $\psi(\mathfrak{g}) \subset \operatorname{rad}(H_\Gamma^{(\infty)})$ and hence also $\psi(\mathfrak{g}) \subset \operatorname{rad}(e_{\Lambda,\Gamma} H_\Lambda^{(\infty)} e_{\Lambda,\Gamma})$.

Now $H_\Gamma^{(\infty)}$ and hence also $e_{\Lambda,\Gamma} H_\Lambda^{(\infty)} e_{\Lambda,\Gamma}$ is a free $\psi(\hat{D})_0$-module and therefore $\psi(\mathfrak{g})$ is generated by a regular sequence in $e_{\Lambda,\Gamma} H_\Lambda^{(\infty)} e_{\Lambda,\Gamma}$. Hence the fact that $H_\Gamma^{(\infty)}$ has finite global dimension together with (9.13) implies that $e_{\Lambda,\Gamma} H_\Lambda^{(\infty)} e_{\Lambda,\Gamma}$ also has finite global dimension.

(9.14) implies that

$$e_{\Lambda,\Gamma} H_\Lambda^{(\infty)} e_{\Lambda,\Gamma} = e_{\Lambda,\Gamma} H_\Lambda^p e_{\Lambda,\Gamma} \mathbin{\hat{\otimes}} k[[(\pi_i)_{i\notin T}]]$$

and hence $e_{\Lambda,\Gamma} H_\Lambda^p e_{\Lambda,\Gamma}$ also has finite global dimension. Now corollary 9.4.5 implies (9.15). □

The faces in Q^p are of the form

$$F = \operatorname{pr}_K^{-1}(v)$$

where $K \subset T$ and $v \in \{-1,0\}^K$. The parallelism class of F is determined by K.

Proposition 9.5.2. *There is a face in $Q^p - S_\Gamma$ in the parallelism class associated to $K \subset T$ if and only if $(\eta_i)_{i\notin K}$ does not span $\mathfrak{g}^*$ as a vector space.*

Proof. The property that $\operatorname{pr}_K^{-1}(v) \not\subset Q^p - S_\Gamma$ for all $v \in \{-1,0\}^K$ means that, whatever the choice of $v \in \{-1,0\}^K$, χ can always be written as $\sum_{i=1}^n \alpha_i\eta_i$, $\alpha \cong \theta \bmod \mathbb{Z}^n$ and

$$\forall i \in K: \begin{array}{l} v_i = 0 \Rightarrow \alpha_i \geq 0 \\ v_i = -1 \Rightarrow \alpha_i < 0 \end{array} \tag{9.16}$$

Now let $\mu = \chi - \sum_{i\notin K} \theta_i\eta_i$. Then $\mu \in \sum_{i=1}^n \mathbb{Z}\eta_i$ and (9.16) is equivalent with the property that μ can always be written as $\sum_{i=1}^n u_i\eta_i$ with $u \in \mathbb{Z}^n$ and

$$\forall i \in K: \begin{array}{l} v_i = 0 \Rightarrow u_i \geq 0 \\ v_i = -1 \Rightarrow u_i < 0 \end{array} \tag{9.17}$$

So now we have to show that this property of μ is equivalent with $(\eta_i)_{i\notin K}$ spanning $\mathfrak{g}^*$.

Assume first that $(\eta_i)_{i\notin K}$ spans $\mathfrak{g}^*$ and fix $v \in \{-1,0\}^T$. Let Z be the semigroup spanned by

$$(\eta_i)_{\substack{i\in K\\ v_i=0}}, (-\eta_i)_{\substack{i\in K\\ v_i=-1}}, (\pm\eta_i)_{i\notin K} \tag{9.18}$$

By hypotheses the elements (9.18) do not lie in some cone in $\mathfrak{g}^*$ and hence Z is in fact *equal* to the group generated by the elements (9.18). Hence $\mu + \sum_{\substack{i\in K\\ v_i=-1}} \eta_i \in Z$ which is exactly what we have to show.

Conversely assume that $(\eta_i)_{i\notin K}$ does *not* span $\mathfrak{g}^*$. We will seek a particular v, violating (9.17).

There exist $\psi \in \mathfrak{g}$ such that for all $i \notin K$ one has $\langle \psi, \eta_i\rangle = 0$ and with one of the following additional properties.

(1) If $\mu \notin \sum_{i\notin K} k\eta_i$ then $\langle \psi, \mu\rangle < 0$.
(2) If $\mu \in \sum_{i\notin K} k\eta_i$ then $\exists : i \in K : \langle \psi, \eta_i\rangle < 0$.

Now we choose $v \in \{-1, 0\}^T$ such that

$$\forall i \in K : \begin{array}{l} \langle \psi, \eta_i\rangle \geq 0 \Rightarrow v_i = 0 \\ \langle \psi, \eta_i\rangle < 0 \Rightarrow v_i = -1 \end{array}$$

Applying $\langle \psi, -\rangle$ to $\mu = \sum_{i=1}^n u_i\eta_i$ yields a contradiction with (9.17). □

Corollary 9.5.3. *$F(Q^p - S_\Gamma)$ depends only on T.*

Theorem 9.5.4. *If B^χ has finite global dimension then χ is maximal.*

Proof. Assume that χ is *not* maximal an choose a maximal $\chi' \to \chi$. By definition $\chi \not\to \chi'$. Hence by Theorem 4.4.4 there exists $\theta' \in V(\mathfrak{g} - \chi'(\mathfrak{g}))$ such that

$$\langle \theta'\rangle_A \cap V(\mathfrak{g} - \chi(\mathfrak{g})) = \emptyset$$

Choose $\theta \cong \theta'$ mod $\mathbb{Z}^n$ such that $\theta \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ and put $\Gamma = \theta + L$, $\Gamma' = \theta' + L$, $L = \operatorname{Supp} B^\chi = \operatorname{Supp} B^{\chi'}$.

By construction $|Q^p - S_\Gamma| > |Q^p - S_{\Gamma'}|$. Since χ' is maximal $B^{\chi'}$ has finite global dimension by Theorem 9.3.9 and hence by Theorem 9.5.1

$$F(Q^p - S_{\Gamma'}) = |Q^p - S_{\Gamma'}|$$

Hence

$$|Q^p - S_\Gamma| > |Q^p - S_{\Gamma'}| = F(Q^p - S_{\Gamma'}) = F(Q^p - S_\Gamma)$$

where the last equality follows from corollary 9.5.3.

Thus $|Q^p - S_\Gamma| \neq F(Q^p - S_\Gamma)$ which by Theorem 9.5.1 implies that B^χ has infinite global dimension. □

10. Finite dimensional representations.

10.1. **Generalities.** Let the notation $A, G, \mathfrak{g}, B^\chi, r, s, n, \ldots$ be as before. In this section we will describe the category of finite dimensional representations of A^G. Our most explicit results will be in the cases where $\dim G$ is one or two dimensional. It turns out that especially the case $\dim G = 2$ has some interesting features which do not occur in higher dimensions.

The focus of this section will be the ring A^G, so we fix notations accordingly. For example $\langle \alpha\rangle$ stands for $\langle \alpha\rangle_{A^G}$ (notation: §3.2 and §4) and $L(\alpha)$ will be the corresponding simple A^G representation.

To enhance readablity of this section there will be some duplication with §9.4 and §9.5. However the reader has to keep in mind that in those sections our main focus was B^χ, so the notation is slightly different.

Fix $\mu \in \mathfrak{t}^*$ and let $\Lambda = \mu + \operatorname{Supp} A$, $\Gamma = \mu + \operatorname{Supp} A^G$. For an arbitrary k-algebra R we will denote by R-fin the category of finite dimensional R-modules. Clearly A^G-fin $\subset \mathcal{O}^{(\infty)}$.

Let $\mathcal{O}^{(p)}_{\Gamma,f}$ be the category of finite dimensional objects in $\mathcal{O}^{(p)}_\Gamma$. As usual, A^G-fin decomposes as a direct sum : $A^G\text{-fin} = \oplus_\Gamma \mathcal{O}^{(\infty)}_{\Gamma,f}$

Our first aim is to describe the finite dimensional *simple* modules in $\mathcal{O}^{(\infty)}_\Gamma$ (or equivalently in $\mathcal{O}^{(1)}_\Gamma$). As in §9.5 we put

$$T = \{1, \dots, r\} \cap \{i \mid \mu_i \in \mathbb{Z}\}, \qquad p = |T|$$

Proposition 10.1.1. (1) *For $\mathcal{O}^{(1)}_\Gamma$ to contain non-zero finite dimensional representations, it is* necessary *that the following condition holds :*

$$\text{The } (\eta_i)_{i \notin T} \text{ are linearly independent} \tag{10.1}$$

(2) *Assume that* (10.1) *holds. Then for $\alpha \in \Gamma$ one has* $\dim L(\alpha) < \infty$ *iff there exist $\psi \in \mathfrak{g} \cap \mathbb{Q}^n$ such that :*
(a) $\langle \psi, \eta_i \rangle = 0$ *iff* $i \notin T$.
(b)

$$\text{for all } i \in T\text{:} \quad \begin{array}{l} \langle \psi, \eta_i \rangle < 0 \Rightarrow \alpha_i \in \mathbb{Z}, \alpha_i \geq 0 \\ \langle \psi, \eta_i \rangle > 0 \Rightarrow \alpha_i \in \mathbb{Z}, \alpha_i < 0 \end{array}$$

Proof. Let $\dim L(\alpha) < \infty$ for $\alpha \in \Gamma$. By Proposition 7.2.4 there exists a pair (ψ, θ), attached to χ, such that $\langle \alpha \rangle = \overline{\langle \alpha \rangle} = S_{\psi,\theta} = \overline{\langle \beta \rangle}$ where β is as in definition 7.2.1(4)(5). Since $|\overline{\langle \beta \rangle}| < \infty$ we also have $\langle \beta \rangle < \infty$ and hence $|\overline{\langle \beta \rangle}| = \langle \beta \rangle$. So $\langle \alpha \rangle = \langle \beta \rangle$. This implies in particular that $\alpha \cong \beta \bmod \mathbb{Z}^n$. Since α and β are in the same $V(\mathfrak{g} - \chi(\mathfrak{g}))$ this implies $\beta \in \Gamma$. Since also $\mu \cong \alpha \bmod \mathbb{Z}^n$ we have $\mu_i \notin \mathbb{Z}$ if $\beta_i \notin \mathbb{Z}$. Then (2)(5) of definition 7.2.1 imply that $\langle \psi, \eta_i \rangle = 0$ iff $i \notin T$.

Now we prove (1). Assume that (10.1) does *not* hold. Then lemma 7.2.3 implies that $\dim S_{\psi,\theta} > 0$. But this contradicts the fact that $S_{\psi,\theta} = \langle \alpha \rangle$ is finite.

Now we prove the $\Rightarrow$ direction of (2). We have already shown above that (2a) holds. Since $\langle \alpha \rangle = \langle \beta \rangle$, (2b) follows directly from Definition 7.2.1.

Finally we prove the converse of (2). Let $\alpha \in \Gamma$ and assume $\psi \in \mathfrak{g} \cap \mathbb{Q}^n$ satisfies (2a)(2b). Put $\theta = \sum_{\langle \psi, \eta_i \rangle = 0} \alpha_i \eta_i$. Then by Proposition 7.2.4 $S_{\psi,\theta} = \overline{\langle \alpha \rangle}$ and by lemma 7.2.3, $S_{\psi,\theta}$ is a finite set of points. Hence $\langle \alpha \rangle$ itself is finite, whence $L(\alpha)$ is finite dimensional. □

Remark 10.1.2. It follows easily from Proposition 10.1.1 that a necessary condition for A^G to have finite dimensional representations is that no η_i is equal to zero. This can also be seen directly. Indeed if $\eta_i = 0$ for some i then A^G contains the Weyl algebra $k[x_i, \partial_i]$, and so has no finite dimensional representations.

Now we will describe the category A^G-fin. To be able to state our next theorem we introduce some more notation. H^p will be as in §9.4. It is the completed path algebra of the quiver Q^p also introduced in §9.4. Recall that the vertices of Q^p are given by elements of $\{-1, 0\}^T$ and index the simple objects in $\mathcal{O}^{(1)}_{\Lambda,A}$.

We now define some subsets of Q^p. S_Γ is the set of all $\{-1, 0\}^T$ that correspond to representations in $\mathcal{O}^{(1)}_\Gamma$. That is $v \in S_\Gamma$ iff there exist $\alpha \in \Gamma$ such that

$$\forall i \in T : \quad \begin{array}{l} v_i = 0 \Rightarrow \alpha_i \geq 0 \\ v_i = -1 \Rightarrow \alpha_i < 0 \end{array}$$

In the notation of the previous sections: $e_{\Lambda,\Gamma} = \sum_{v\in S_\Gamma} e_v$ (see §9.5).

By $S_{f,\Gamma}$ we denote the subset of S_Γ whose elements correspond to finite dimensional objects in $\mathcal{O}^{(1)}_\Gamma$. We also write $e_{f,\Gamma} = \sum_{v\in S_{\Gamma,f}} e_v$. S_f will be the subset of all $v \in Q^p$ such that there exist $\psi \in \mathfrak{g} \cap \mathbb{Q}^n$ with the property

$$\begin{aligned} &\forall i \notin T : \langle \psi, \eta_i\rangle = 0 \\ &\forall i \in T : \begin{array}{l} v_i = 0 \Rightarrow \langle \psi, \eta_i\rangle < 0 \\ v_i = -1 \Rightarrow \langle \psi, \eta_i > 0 \end{array} \end{aligned} \tag{10.2}$$

Clearly $S_{f,\Gamma} = S_f \cap S_\Gamma$. Put $e_f = \sum_{v\in S_f} e_v$.

With regard to these subsets of Q^p we will need the following lemma.

Lemma 10.1.3. *For $v \in Q^p$ define v^{opp} by*

$$v^{\text{opp}} = \begin{cases} 0 & \text{if } v_i = -1 \\ -1 & \text{if } v_i = 0 \end{cases}$$

Then if $v \in S_f$, then also $v^{\text{opp}} \in S_f$. However v and v^{opp} cannot both belong to S_Γ.

Proof. Left to the reader. □

Now we define

$$H^p_f = H^p/H^p(1 - e_f)H^p$$

and we have the following result :

Theorem 10.1.4. *Assume that* (10.1) *holds. Then the category $\mathcal{O}^{(\infty)}_{\Gamma,f}$ is equivalent to the category of finite dimensional representations of the algebra*

$$e_{f,\Gamma} H^p_f e_{f,\Gamma} \hat{\otimes} k[[(\pi_i)_{i\notin T}]] \tag{10.3}$$

If $\chi \in \mathfrak{g}^*$ then a similar statement holds for $\mathcal{O}^{(\infty)}_{\Gamma,B^\chi,f}$ but we have to replace (10.3) *by*

$$(e_{f,\Gamma} H^p_f e_{f,\Gamma} \hat{\otimes} k[[(\pi_i)_{i\notin T}]])/(\psi(\mathfrak{g}))$$

where ψ is as in §3.5 and §6.

Proof. We will give the proof for $\mathcal{O}^{(\infty)}_{\Gamma,f}$. The proof for $\mathcal{O}^{(\infty)}_{\Gamma,B^\chi,f}$ is completely similar.

An object in $\mathcal{O}^{(\infty)}_\Gamma$ is in $\mathcal{O}^{(\infty)}_{\Gamma,f}$ if it has no infinite dimensional composition factors. So from

$$H^{(\infty)}_\Gamma = e_{\Lambda,\Gamma} H^{(\infty)}_\Lambda e_{\Lambda,\Gamma}$$

(Prop. 4.3.1(3)), together with

$$H^{(\infty)}_\Lambda = H^p \hat{\otimes} k[[(\pi_i)_{i\notin T}]]$$

we obtain that $\mathcal{O}^{(\infty)}_{\Gamma,f}$ is equivalent with the category of finite dimensional representations of

$$\frac{e_{\Lambda,\Gamma} H^p e_{\Lambda,\Gamma}}{e_{\Lambda,\Gamma} H^p (e_{\Lambda,\Gamma} - e_{f,\Gamma}) H^p e_{\Lambda,\Gamma}} \hat{\otimes} k[[(\pi_i)_{i\notin T}]]$$

We have to show that this is equal to (10.3). To prove this we first prove the following claim :

$$(1 - e_f)H^p e_{\Lambda,\Gamma} \subset H^p(e_{\Lambda,\Gamma} - e_{f,\Gamma})H^p \tag{10.4}$$

Note that the left side of (10.4) is topologically spanned by reduced paths starting in S_Γ and ending in the complement in S_f.

So let x be a reduced path starting in $v \in S_\Gamma$ and ending in $w \notin S_f$. We will show that x is in the right side of (10.4). The fact that $w \notin S_f$ means that there exist $(\gamma_i)_i \in \mathbb{Z}^n$ such that $\sum \gamma_i \eta_i = 0$ and

$$\forall i \in T : \begin{array}{c} w_i = 0 \Rightarrow \gamma_i \geq 0 \\ w_i = -1 \Rightarrow \gamma_i \leq 0 \end{array} \tag{10.5}$$

and such that there is at least one $i \in T$ with $\gamma_i \neq 0$. The fact that $v \in S_\Gamma$ means that there exists $\alpha \in \Gamma$ such that

$$\forall i \in T : \begin{array}{c} v_i = 0 \Rightarrow \alpha_i \geq 0 \\ v_i = -1 \Rightarrow \alpha_i < 0 \end{array} \tag{10.6}$$

Now we define $v' \in Q^p$ as follows :

$$v'_i = \begin{cases} w_i & \text{if } \gamma_i \neq 0 \\ v_i & \text{otherwise} \end{cases}$$

Put $\alpha' = \alpha + N\gamma$, $N \in \mathbb{N}$, $N \gg 0$. Then α' satisfies (10.6) if we replace v by v'. So $v' \in S_\Gamma$. We claim that also $v' \notin S_f$. Assume the contrary. Then there exist $\psi \in \mathfrak{g} \cap \mathbb{Q}^n$ such that

$$\begin{gathered} \forall i \notin T : \langle \psi, \eta_i \rangle = 0 \\ \forall i \in T : \begin{array}{c} v'_i = 0 \Rightarrow \langle \psi, \eta_i \rangle < 0 \\ v'_i = -1 \Rightarrow \langle \psi, \eta_i \rangle > 0 \end{array} \end{gathered} \tag{10.7}$$

Applying $\langle \psi, - \rangle$ to $\sum_i \gamma_i \eta_i$ yields $\sum_{i \in T} \gamma_i \langle \psi, \eta_i \rangle = 0$.

Now if $\gamma_i \neq 0$ for $i \in T$, then v'_i and w_i are equal, and comparing (10.5) with (10.7) we see that if $\gamma_i \neq 0$ then $\gamma_i \langle \psi, \eta_i \rangle < 0$. Since at least one $(\gamma_i)_{i \in T}$ is non-zero, this yields a contradiction.

Now clearly there exists a reduced path x' form v to w passing through v'. Hence $x = x'$ belongs to $H^p(e_{\Lambda,\Gamma} - e_{f,\Lambda})H^p$. This finishes the proof of (10.4).

From (10.4) we deduce the inclusion

$$e_{\Lambda,\Gamma} H^p (1 - e_f) H^p e_{\Lambda,\Gamma} \subset e_{\Lambda,\Gamma} H^p (e_{\Lambda,\Gamma} - e_{f,\Gamma}) H^p e_{\Lambda,\Gamma}$$

This is in fact an equality since the opposite inclusion follows trivially from $e_{f,\Gamma} = e_f e_{\Lambda,\Gamma}$.

So we obtain

$$\frac{e_{\Lambda,\Gamma} H^p e_{\Lambda,\Gamma}}{e_{\Lambda,\Gamma} H^p (e_{\Lambda,\Gamma} - e_{f,\Gamma}) H^p e_{\Lambda,\Gamma}} = \frac{e_{\Lambda,\Gamma} H^p e_{\Lambda,\Gamma}}{e_{\Lambda,\Gamma} H^p (1 - e_f) H^p e_{\Lambda,\Gamma}} = e_{\Lambda,\Gamma} H^p_f e_{\Lambda,\Gamma}$$

Since $e_{\Lambda,\Gamma} H^p_f e_{\Lambda,\Gamma}$ is annihilated by $1 - e_f$, on the left and on the right, it is equal to $e_f e_{\Lambda,\Gamma} H^p_f e_{\Lambda,\Gamma} e_f = e_{f,\Gamma} H^p_f e_{f,\Gamma}$. This finishes the proof of the theorem. □

Remark 10.1.5. Part of the usefulness of Theorem 10.1.4 stems from the fact that H^p_f can be described as the completed path algebra of the full subquiver of Q^p_f of Q^p having the set S_f as vertices (thus an arrow $v \to w$ in Q^p is in Q^p_f iff $v, w \in S_f$). The relations on Q^p_f are deduced in a trivial way from those of on Q^p. That is if $u, v, w \in S_f$, $v' \in Q^p$ are such that $v \neq v'$, $u \neq w$ and the arrows $u \to v \to w$, $u \to v' \to w$ are defined in Q^p then

$$u \to v \to w = \begin{cases} u \to v' \to w & \text{if } v' \in S_f \\ 0 & \text{otherwise} \end{cases}$$

Theorem 10.1.4 together with the foregoing remark yield the following general result.

Corollary 10.1.6. *Assume that for all $i \in \{1, \ldots, r\}$ there exists a $j \in \{1, \ldots, r\}$, $j \neq i$ such that η_i and η_j are proportional in $X(G)_{\mathbb{Q}}$. Then A^G-fin is semisimple.*

Proof. If there is some $\eta_i = 0$ then by remark 10.1.2, A^G-fin only contains the zero-representation. So in that case the corollary is true. Hence assume $\eta_i \neq 0$ for all i.

Assume that $\mathcal{O}_\Gamma^{(\infty)}$ contains a non-trivial finite dimensional representation. By Proposition 10.1.1(1), if η_i and η_j are proportional then they are not both contained in the complement of T. However if $\psi \in \mathfrak{g} \cap \mathbb{Q}^n$ then $\langle \psi, \eta_i \rangle = 0$ implies $\langle \psi, \eta_j \rangle = 0$. So by Proposition 10.1.1(2a) we find that $\{\eta_i, \eta_j\} \subset T$. Hence $\{1, \ldots, r\} = T$. Furthermore the signs of $\langle \psi, \eta_i \rangle$ and $\langle \psi, \eta_j \rangle$ mutually determine each other. Hence no two distinct vertices in S_f can be adjacent in Q^p. So $H_f^p = \oplus_{v \in S_f} k e_v$ is semi-simple and hence the same is true for $\mathcal{O}_{\Gamma,f}^\infty$ by Theorem 10.1.4. $\square$

10.2. $\dim \mathfrak{g} = 1$**.** Corollary 10.1.6 applies almost immediately to the case $\dim \mathfrak{g} = 1$. Since we we are interested in non-trivial cases, we may assume by remark 10.1.2 that $\eta_i \neq 0$ for all i. Furthermore the case $n = 1$ is somewhat special. In that case $A^G = k[\pi]$ and the reader may verify that some of the assertions in Proposition 10.2.1 below are false in that case. So we assume $n > 1$.

Finally if $s > 0$ then it follows from lemma 9.3.6 together with [15] that $B^\chi = A^G/(\mathfrak{g} - \chi(\mathfrak{g}))$ is simple, whence A^G has no finite dimensional representation. So again to avoid trivialities, we take $s = 0$.

Assuming all these conditions we have the following result which is very reminiscent of what happens in the case of $U(\mathfrak{sl}_2)$. As usual we identify $X(G) = Y(G) = \mathbb{Z}$.

Proposition 10.2.1. *Assume that* $\dim G = 1$*,* $n > 1$*,* $s = 0$ *and* $\eta_i \neq 0$ *for all* i*. Then*

1. *For all $\chi \in \mathfrak{g}^*$, B^χ has at most one finite dimensional simple representation.*
2. *B^χ has a finite dimensional representation if and only if there exist $a_1, \ldots, a_n \in \mathbb{Z}$ such that $\sum a_i \eta_i = \chi$ and one of the following holds*
$$\forall i \in \{1, \ldots, n\} : \begin{array}{l} \eta_i < 0 \Rightarrow a_i \geq 0 \\ \eta_i > 0 \Rightarrow a_i < 0 \end{array}$$
or
$$\forall i \in \{1, \ldots, n\} : \begin{array}{l} \eta_i < 0 \Rightarrow a_i < 0 \\ \eta_i > 0 \Rightarrow a_i \geq 0 \end{array}$$
3. *The category of finite dimensional representations of A^G is semi-simple.*

Proof. Let $\chi \in \mathfrak{g}^*$ and choose $\mu \in V(\mathfrak{g} - \chi(\mathfrak{g}))$. Put $\Gamma = \mu + \operatorname{Supp} A^G$. Assume that $L(\alpha) \in \mathcal{O}_\Gamma^{(1)}$ for some $\alpha \in \Gamma$. By (10.1) we see that the complement of T can at most contain one element. Let ψ be as in Proposition 10.1.1. Assume $i \notin T$ and note that for all $\psi \in \mathfrak{g} \cap \mathbb{Q}^n \setminus \{0\}$ one has $\langle \psi, \eta_i \rangle \neq 0$. Hence applying (2a) of Proposition 10.1.1 we have $\psi = 0$. But then applying 10.1.1(2a) again for $\psi = 0$ we obtain $T = \emptyset$. So $n \leq 1$, which contradicts the hypotheses. Therefore $T = \{1, \ldots, n\}$ and in particular $\mu \in \mathbb{Z}^n$. This implies that Γ is uniquely determined since if $\mu_1, \mu_2 \in \mathbb{Z}^n$ are such that $\mu_i + \operatorname{Supp} A^G \subset V(\mathfrak{g} - \chi(\mathfrak{g}))$ for $i = 1, 2$ then $\mu_1 - \mu_2 = \mathbb{Z}^n \cap V(\mathfrak{g}) = \operatorname{Supp} A^G$.

Now Proposition 10.1.1(2a) implies that $\psi \neq 0$. Hence allowing α to vary there are at most two possible choices for ψ say ± 1 (up to positive scalar multiples). Now the signs of the components of α are determined by ψ. So there can be at most two vertices in S_Γ corresponding to finite dimensional representations. It is clear that these two vertices must be opposite to each other in Q^p. But lemma 10.1.3 implies that opposite vertices cannot both belong to S_Γ.

Hence there can be at most one finite dimensional simple representation in $\mathcal{O}_\Gamma^{(1)}$. Since $\Gamma \in V(\mathfrak{g} - \chi(\mathfrak{g}))$ was itself unique, with respect to the property of containing finite dimensional representations, we deduce that B^χ has at most one finite dimensional simple representatation. This proves (1). To prove (2) we observe that for B^χ to have finite dimensional representations there have to exist $\alpha \in V(\mathfrak{g} - \chi(\mathfrak{g}))$, $\psi \in \mathfrak{g} \cap \mathbb{Q}^n$ satisfying (2b) of Proposition 10.1.1. In the proof of (1) above we have already seen that we may take $\alpha \in \mathbb{Z}^n$, $\psi = \pm 1$. Plugging this into 10.1.1(2b) we find exactly the condition stated under (2) of the current proposition.

(3) follows immediately from corollary 10.1.6. □

10.3. $\dim \mathfrak{g} = 2$. In this case we can still give a fairly explicit description of the category B^χ-fin for $\chi \in \mathfrak{g}^*$. Our result is as follows :

Proposition 10.3.1. *Assume that* $\dim G = 2$. *Then* B^χ-fin *is equivalent to the category of finite dimensional representations over a quiver which is a finite (perhaps empty) union of quivers of the type*

$$\begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] & \cdots & \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \end{tikzcd}$$

with relations given by all paths of length 2. In particular B^χ-fin *has finite representation type.*

Proof. The fact that the indicated quiver has finite representation type follows from [34, Prop. 2.3]. So we only have to prove the first part of the proposition. As usual we can make a few reductions. To start we assume $\eta_i \neq 0$ for all i since otherwise A^G has no finite dimensional representations and there is nothing to prove. Furthermore we assume that the weights $(\eta_i)_i$ generate $X(G)_\mathbb{Q}$ rationally since otherwise we could reduce to the case $\dim \mathfrak{g} = 1$. In that case B^χ-fin is semisimple and contains at most a unique simple finite dimensional representation, and we are done. Also, if $n \leq 2$ then, assuming the earlier conditions, we obtain $B^\chi = k$ and again there is nothing to do. So we assume $n > 2$.

Since B^χ has only a finite number of primitive ideals, it has in particular only a finite number of finite dimensional simple representations. Hence there are only a finite number of $\Gamma \subset V(\mathfrak{g} - \chi(\mathfrak{g}))$ with the property that $\mathcal{O}_\Gamma^{(1)}$ contains nontrivial representations. Hence it is sufficient to prove the current proposition for an individual $\mathcal{O}_\Gamma^{(1)}$.

So fix one particular Γ such that $\mathcal{O}_\Gamma^{(1)}$ contains finite dimensional representations. Then (10.1) implies that the complement of T contains at most two elements. We analyze the different possibilities.

- $|T^c| = 2$. Condition 10.1.1(2a) now implies $\psi = 0$, and applying 10.1.1(2a) again for $\psi = 0$ we find $T = \emptyset$. Hence $n = 2$, which was excluded in the beginning of the proof.
- $|T^c| = 1$. By hypotheses $p = |T| \geq 2$. Condition (2a) of Proposition 10.1.1 shows that, with respect to the signs of $\langle \psi, \eta_i \rangle$ there are two non-equivalent

ψ's, say $\pm\psi_1$. Hence Q_f^p consists of two elements which are opposite, and since $p \geq 2$ one sees that these are non-adjacent.

Thus H_f^p is semi-simple. So by Theorem 10.1.4, $\mathcal{O}_{\Gamma,f}^{(\infty)}$ is also semi-simple.

- $|T^c| = 0$. This is the most interesting case. Now 10.1.1(2a) implies $\langle\psi, \eta_j\rangle \neq 0$ for all ψ. By ordering the essentially different ψ's in counter clockwise sense around the origin one easily sees that Q_f^p is either

$$\begin{tikzcd} \overset{\bullet}{v_1} \arrow[r, shift left, "X_1"] & \overset{\bullet}{v_2} \arrow[l, shift left, "Y_1"] \arrow[r, shift left, "X_2"] & \overset{\bullet}{v_3} \arrow[l, shift left, "Y_2"] & \cdots & \overset{\bullet}{v_{2n}} \arrow[r, shift left, "X_{2n}"] & \overset{\bullet}{v_1} \arrow[l, shift left, "Y_{2n}"] \end{tikzcd} \tag{10.8}$$

(the first and the last vertex are identified) or else is a finite union of quivers of the form

$$\begin{tikzcd} \overset{\bullet}{v_1} \arrow[r, shift left, "X_1"] & \overset{\bullet}{v_2} \arrow[l, shift left, "Y_1"] \arrow[r, shift left, "X_2"] & \overset{\bullet}{v_3} \arrow[l, shift left, "Y_2"] & \cdots & \overset{\bullet}{v_t} \arrow[r, shift left, "X_t"] & \overset{\bullet}{v_{t+1}} \arrow[l, shift left, "Y_t"] \end{tikzcd} \tag{10.9}$$

(this case occurs when some of the η_i are proportional) where the relations are given by

$$X_{i+1}X_i = 0 \qquad \text{and} \qquad Y_iY_{i+1} = 0$$

(in (10.8) we take $X_{2n+1} = X_1$, $Y_{2n+1} = Y_1$. This convention remains in force below.)

Now we compute $H_f^p/\psi(\mathfrak{g})$. In (10.8) and (10.9) we have

$$Y_iX_i = e_{v_i}\pi_{k_i}$$
$$X_iY_i = e_{v_{i+1}}\pi_{k_i}$$

for some $\pi_{k_i} \in \{\pi_1, \dots, \pi_n\}$.

Choose a basis $\{f_1, f_2\}$ for $\mathfrak{g}$ and use this basis to identify $\mathfrak{g}^*$ with k^2. Then we may write $\eta_i = (\eta_{i1}, \eta_{i2}) \in k^2$ and $\psi(f_j) = \sum_i \eta_{ij}\pi_i$.

Then in case (10.8) $H_f^p/(\psi(\mathfrak{g})$ is the completed path algebra of the same quiver but with additional relations

$$\eta_{k_{i+1}j}Y_{i+1}X_{i+1} + \eta_{k_ij}X_iY_i = 0 \qquad (j = 1, 2,\ i = 1, \dots, 2n) \tag{10.10}$$

In case (10.9) $H_f^p/(\psi(\mathfrak{g})$ is a completed path algebra of the quiver given by (10.9) but with additional relations

$$\eta_{k_{i+1}j}Y_{i+1}X_{i+1} + \eta_{k_ij}X_iY_i = 0 \qquad (j = 1, 2,\ i = 1, \dots, n) \tag{10.11}$$

$$\eta_{k_1j}Y_1X_1 = 0 \qquad (j = 1, 2) \tag{10.12}$$

$$\eta_{k_tj}X_tY_t = 0 \qquad (j = 1, 2) \tag{10.13}$$

Now for a vertex v_i for which (10.10) or (10.11) applies we have that η_{k_i}, $\eta_{k_{i+1}}$ are not proportional and hence we obtain the additional relations $X_iY_i = 0$, $Y_{i+1}X_{i+1} = 0$. Similarly, have assumed $\eta_i \neq 0$ for all i and so (10.12) (10.13) imply $Y_1X_1 = 0$, $X_tY_t = 0$.

So we obtain that $H_f^p/(\psi(\mathfrak{g})$ is the completed path algebra of either (10.8) or (10.9) with relations given by all paths of length 2. However one immediately sees that the completion is unnecessary because the path algebra modulo the relations is finite dimensional.

Now to finish the proof we invoke Theorem 10.1.4. This amount to picking out of the quivers (10.8) and (10.9) the vertices that belong to S_Γ.

The only bad case that might occur is that we end up with a quiver of the form (10.8). This would imply $S_f = S_\Gamma$. However this is impossible since by lemma 10.1.3 for an arbitrary vertex v, v and v^{opp} cannot both belong to S_Γ. □

Remark 10.3.2. It follows from Proposition 10.2.1 that B^χ-fin $\subset \mathcal{O}_\Gamma^{(1)}$ if $\dim G = 2$. This is not true in higher dimension. Likewise the fact that B^χ-fin has finite representation type does not generalize to higher dimension.

11. An example

In this section we apply the foregoing results to an explicit example which may be considered typical. We take

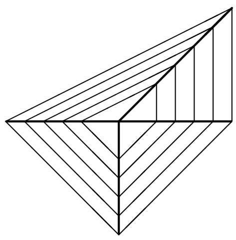
$$A = k[x_1, x_2, x_3, x_4, \partial_1, \partial_2, \partial_3, \partial_4]$$

and G will be a two-dimensional torus acting with weights $\eta_{1,\dots,4}$ on $x_{1,\dots,4}$ respectively. We identify $X(G)$ with $\mathbb{Z}^2$ and we assume that $\eta_1, \eta_2, \eta_3, \eta_4$ are given by $(1,0)$, $(1,0)$, $(0,1)$, $(-1,1)$. As before, for $\chi \in \mathfrak{g}^*$, we put

$$B^\chi = A^{\mathfrak{g}}/(\mathfrak{g} - \chi(\mathfrak{g}))$$

B^χ is a domain of GK dimension 4 and Krull dimension 2 (see §8). Below we will compute the χ's for which B^χ is simple or has finite global dimension. We will also, for each χ describe, the lattice of primitive ideals of B^χ and we will give the $\to$-relation between different χ's. The most interesting case is when $\chi \in X(G) = \mathbb{Z}^2$. In that case there are 15 equivalence classes which are related as in figure 11.3 below (we call χ, χ' *equivalent* if $\chi \to \chi'$ and $\chi' \to \chi$ both hold).

Remark 11.1. It follows from [19] that in this case B^χ is a ring of twisted differential operators (tdo) on the first Hirzebruch surface. Recall that the first Hirzebruch surface is given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. Alternatively, it can be considered as a toric variety with fan



By taking we $\eta_4 = (-n, 1)$, $n \in \mathbb{N}$ we could also have treated the other Hirzebruch surfaces. However the result are completely analogous to those for the first one.

Recall that the primitive ideals of B^χ are indexed by pairs (ψ, θ) satisfying the conditions of definition 7.2.1(1)..(5) (see remark 7.3.2(1) and Proposition 7.7.1(2)). All these pairs fit together in a single large poset $\mathcal{P}$ (see §7.7)

The identification $X(G) = \mathbb{Z}^2$ allows us to identify $\mathfrak{g} = \mathfrak{g}^* = k^2$. Let Ξ be an equivalence class for the comparability relation on $\mathfrak{g} = k^2$ (see §4.4). By Proposition 7.6.1 Ξ is an element of $k^2/\mathbb{Z}^2$.

Define

$$\mathcal{P}_\Xi = \bigcup_{\chi \in \Xi} \mathcal{P}_\chi$$

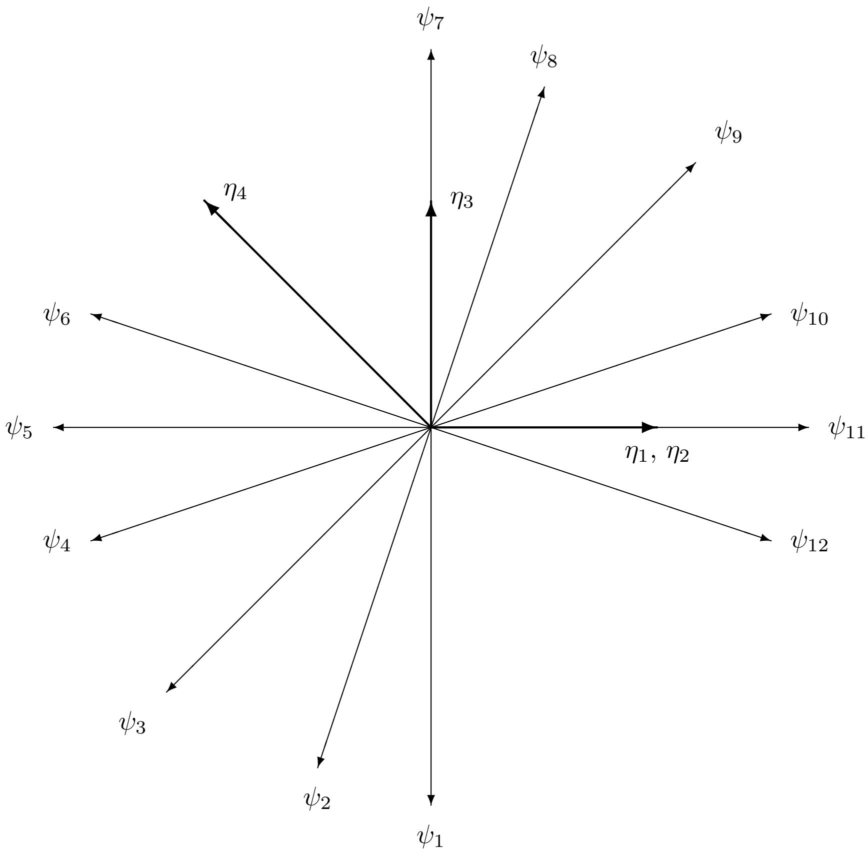


FIGURE 11.1. The essentially different ψ's

Then

$$\mathcal{P} = \bigcup_{\Xi} \mathcal{P}_\Xi$$

and further the elements of different $\mathcal{P}_\Xi$ are incomparable under the ordering on $\mathcal{P}$. Below we will describe the posets $\mathcal{P}_\Xi$ as Ξ varies.

Let us first give in Figure 11.1 the essentially different non-zero ψ's. We also let $\psi_0 = 0$. Since all subsets of $(\eta_i)_i$ span a direct summand of $\mathbb{Z}^2$ we have for all ψ

$$\left(\sum_{i=1}^{4} \mathbb{Z}\eta_i\right) \cap \left(\sum_{\langle\psi,\eta_i\rangle=0} k\eta_i\right) = \sum_{\langle\psi,\eta_i\rangle=0} \mathbb{Z}\eta_i \tag{11.1}$$

If (ψ, θ_1) is attached to χ_1 and (ψ, θ_2) is attached to χ_2 with χ_1, χ_2 comparable then (4) and (5) of definition 7.2.1 imply that $\chi_1 - \theta_1 \in \sum \mathbb{Z}\eta_i$, $\chi_2 - \theta_2 \in \sum \mathbb{Z}\eta_i$

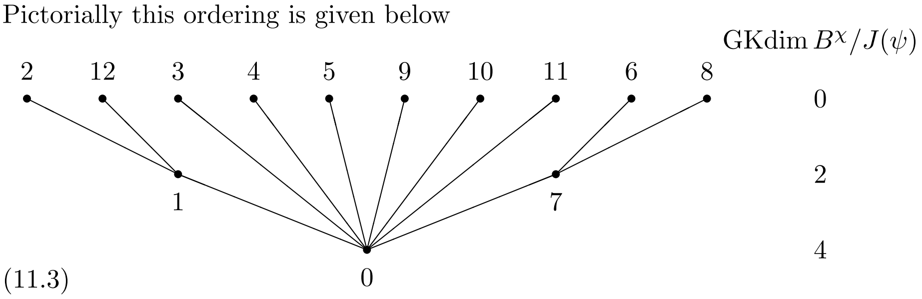
and hence

$$\theta_1 - \theta_2 \in \left(\sum_{i=1}^{4} \mathbb{Z}\eta_i \right) \cap \left(\sum_{\langle \psi, \eta_i \rangle = 0} k\eta_i \right) = \sum_{\langle \psi, \eta_i \rangle = 0} \mathbb{Z}\eta_i$$

Hence if (ψ, θ) is attached to χ then θ is uniquely determined by the comparability class of χ (in contrast with example 7.2.7 where (11.1) didn't hold). So in the sequel we will say that ψ is attached to χ if there exists a θ (necessarily unique) such that (ψ, θ) is attached to χ. Furthermore the ordering on $\mathcal{P}$ as defined in §7.7, when restricted to $\mathcal{P}_\Xi$, simplifies to

$$\psi \geq \psi' \quad \text{iff} \quad (11.2) \quad \begin{array}{l} \{i \mid \langle \psi', \eta_i \rangle < 0\} \subset \{i \mid \langle \psi, \eta_i \rangle < 0\} \\ \{i \mid \langle \psi', \eta_i \rangle > 0\} \subset \{i \mid \langle \psi, \eta_i \rangle > 0\} \end{array}$$

Pictorially this ordering is given below



(11.3)

Here we have written $J(\psi)$ for the primitive ideal $J(\psi, \theta)_{B^\chi}$ of B^χ, introduced in §7.7. The GK-dimension of the primitive quotient $B^\chi/J(\psi)$ is computed with the formula

$$\operatorname{GKdim} B^\chi/J(\psi, \theta)_{B^\chi} = 2 \dim S_{\psi,\theta}$$

which follows from corollary 8.2.2 together with Proposition 7.2.4. $\dim S_{\psi,\theta}$ can be computed using lemma 7.2.3.

Remark 11.2. Similarly to what one does for enveloping algebras one can define, for a given pair (ψ, θ), the function $\nu(\chi) = \operatorname{Goldie\,rk}(B^\chi/J(\psi, \theta)_{B^\chi})$. ν is defined on those χ for which (ψ, θ) is attached to χ. By corollary 7.4.3 $\nu(\chi)$ is the number of connected components of $S_{\psi,\theta}$ (which depends on χ). Using this fact one can easily compute ν for $\psi_1, \ldots, \psi_{12}$. We do not list the results since they are not very illuminating. Let us suffice by saying that we obtain polynomials of degree 0 for ψ_0, of degree 1 for ψ_1, ψ_7 and of degree 2 for the other ψ's. The relation with the GK-dimension of $B^\chi/J(\psi)$ is clear.

The fact that we obtain polynomials is a feature of this example and does not hold in general. However by suitably extending the notion of degree it is possible to generalize the connection with GK dimension.

Table 11.1 lists the χ's that are attached to the various ψ_i's. This amounts to verifying definition 7.2.1(5). Using the identification $\mathfrak{g}^* = k^2$, we have written χ as a pair $(\chi_1, \chi_2) \in k^2$. Inspection of this table reveals that it is natural to separate the χ's into five disjoint families, each closed under comparability.

(A) $\chi_1 \notin \mathbb{Z}$, $\chi_2 \notin \mathbb{Z}$, $\chi_1 + \chi_2 \notin \mathbb{Z}$.
(B) $\chi_1 \in \mathbb{Z}$, $\chi \notin \mathbb{Z}^2$.
(C) $\chi_2 \in \mathbb{Z}$, $\chi \notin \mathbb{Z}^2$.
(D) $\chi_1 + \chi_2 \in \mathbb{Z}$, $\chi \notin \mathbb{Z}^2$.
(E) $\chi \in \mathbb{Z}^2$

ψ	χ
0	no condition
1	$\chi_2 \in \mathbb{N}$
2	$\chi \in \mathbb{N}(1,0)+\mathbb{N}(-1,1)$
3	$\chi_1+\chi_2 \in \mathbb{N}, \quad \chi \notin \mathbb{Z}^2$
4	$\chi \in (1,-1)+\mathbb{N}(0,1)+\mathbb{N}(1,-1)$
5	$\chi_1 \in 1+\mathbb{N}, \quad \chi \notin \mathbb{Z}^2$
6	$\chi \in (1,-2)+\mathbb{N}(1,0)+\mathbb{N}(0,-1)$
7	$\chi_2 \in -2-\mathbb{N}$
8	$(-1,-2)+\mathbb{N}(-1,0)+\mathbb{N}(1,-1)$
9	$\chi_1+\chi_2 \in -3-\mathbb{N}, \quad \chi \notin \mathbb{Z}^2$
10	$\chi \in (-2,-1)+\mathbb{N}(-1,1)+\mathbb{N}(0,-1)$
11	$\chi_1 \in -2-\mathbb{N}, \quad \chi \notin \mathbb{Z}^2$
12	$\chi \in (-2,0)+\mathbb{N}(-1,0)+\mathbb{N}(0,1)$

TABLE 11.1. The χ's attached to a given ψ

We will analyze these families separately. Let us first recapitulate some of the facts we will need. The injective dimension of B^χ is given by the following formula

$$\operatorname{inj\,dim} B^\chi = 4 - \frac{1}{2} \min_{\psi \text{ att. to } \chi} \operatorname{GKdim} B^\chi/J(\psi) \tag{11.4}$$

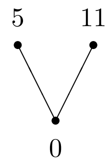
which follows by combining Theorem 8.4.1(3) with corollary 8.2.2. Note hereby that $\operatorname{GKdim} B^\chi/J(\psi)$ was already given in (11.3). Recall also that $\operatorname{inj\,dim} B^\chi = \operatorname{gl\,dim} B^\chi$ if the latter is finite (lemma 9.1.2). By Theorem 9.1.1 $\operatorname{gl\,dim} B^\chi$ is finite iff χ is maximal.

Below Ξ will stand for an equivalence class for the comparability relation. We recall that the $\rightarrow$-relation on Ξ may be deduced from Proposition 7.7.1(3).

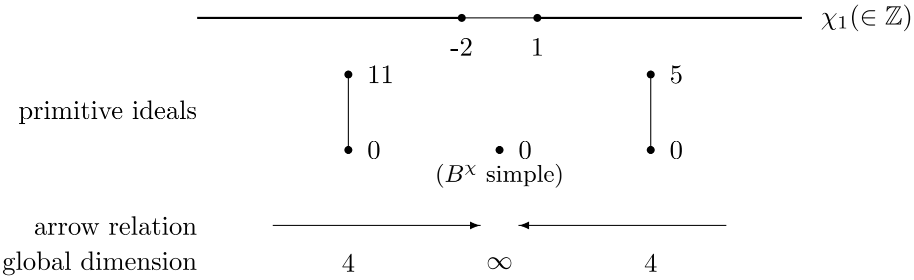
Family (A). In this case only ψ_0 is attached to the elements of Ξ. So by Proposition 7.7.1(3) all B^χ for $\chi \in \Xi$ are Morita equivalent. In particular every $\chi \in \Xi$ is both minimal and maximal and so B^χ is simple and has finite global dimension. By (11.4) we obtain

$$\operatorname{gl\,dim} B^\chi = 2$$

Family (B). Here ψ_0, ψ_5 and ψ_{11} are attached to members of Ξ. By restriction from (11.3) we obtain for $\mathcal{P}_\Xi$:

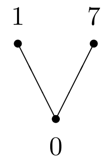


The behavior of the B^χ for $\chi \in \Xi$ may be graphically represented as follows :

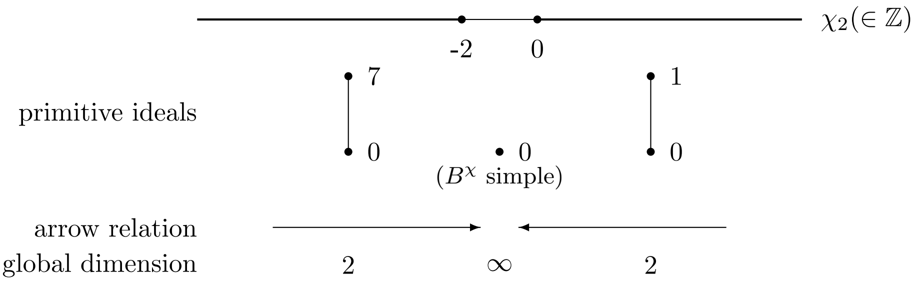


Note that this picture is very similar to the situation for a one dimensional torus (see [28]).

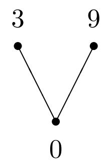
Family (C). Now ϕ_0, ϕ_1 and ψ_7 are attached to members of Ξ. By restriction from (11.3) we obtain $\mathcal{P}_\Xi$:



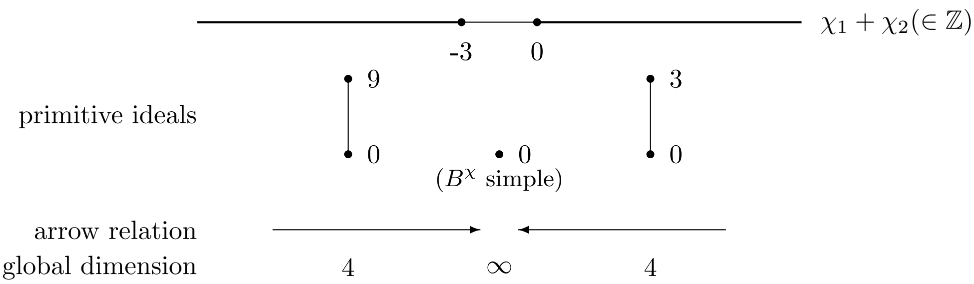
The behavior of the B^χ for $\chi \in \Xi$ may be graphically represented as follows :



Family (D). Now ϕ_0, ϕ_3 and ψ_9 are attached to members of Ξ. By restriction from (11.3) we obtain $\mathcal{P}_\Xi$:



The behavior of the B^χ for $\chi \in \Xi$ may be graphically represented as follows :



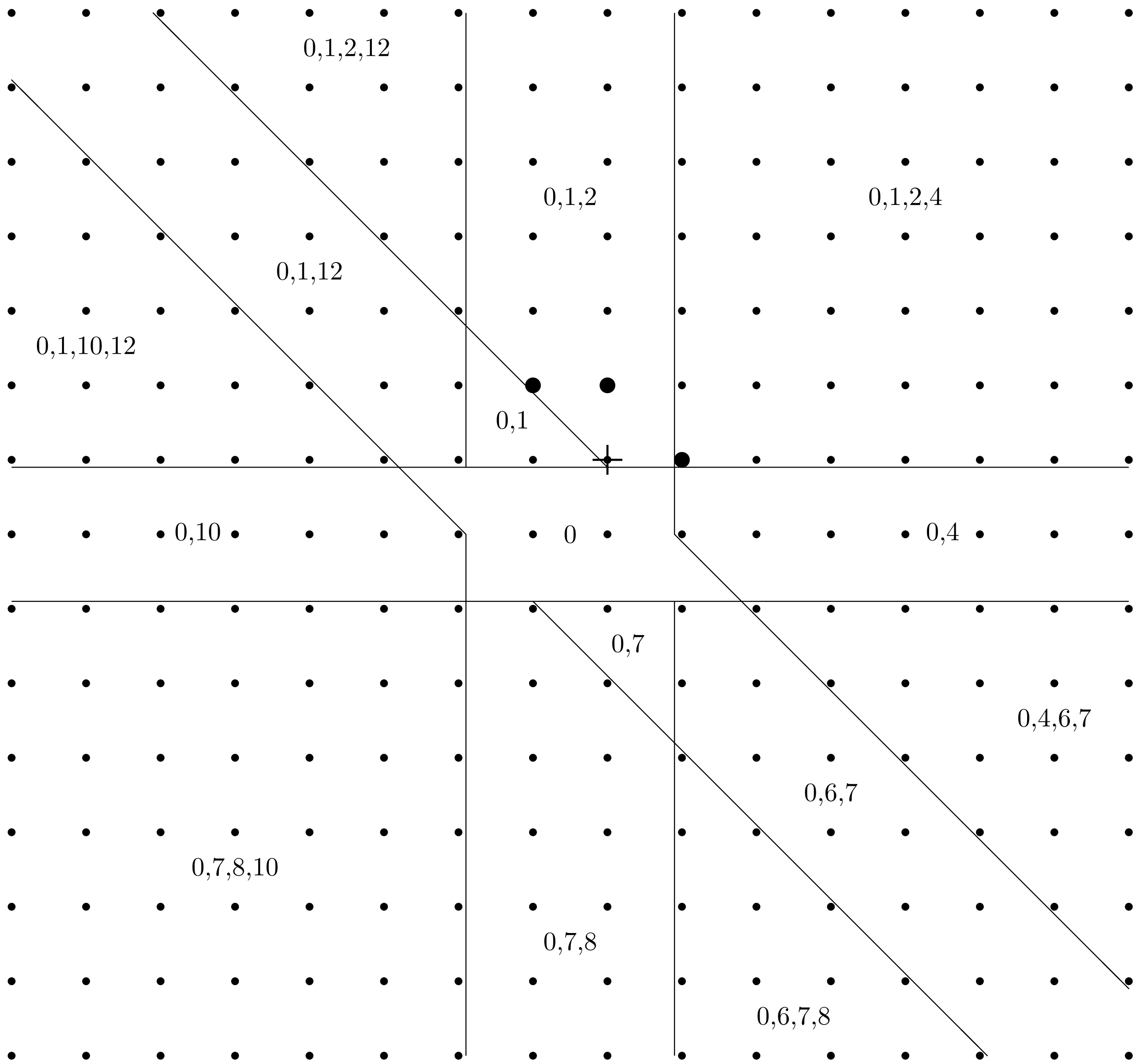
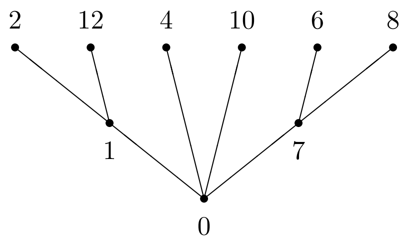


FIGURE 11.2. The ψ's attached to a given χ.

Family (E). This is the most interesting case. Now we have for $\mathcal{P}_{\Xi}$



The ψ's that are attached to the various χ's are represented graphically in Figure 11.2. We deduce that there are fifteen distinct equivalence classes in Ξ. Their lattices of primitive ideals together with the $\to$-relation are given in Figure 11.3.

The B^χ's corresponding to fat dots have finite global dimension. The central B^χ is simple. The B^χ's on the exterior have injective dimension 4. The central one has injective dimension 2 and the intermediate ones have injective dimension 3.

To finish this example let us determine the category of finite dimensional representations of B^χ, where χ belongs to one of the families we have defined. To this end we recall the strategy that was exhibited in Section 10. First one determines, with the help of Proposition 10.1.1, those $\Gamma \subset V(\mathfrak{g} - \chi(\mathfrak{g}))$ for which $\mathcal{O}_\Gamma^{(1)}$ contains finite dimensional representations. Then for the individual Γ's one uses Theorem 10.1.4. Recall that the first step in applying Theorem 10.1.4 consists in determining the quiver Q_f^p. Since in our example one has $\dim \mathfrak{g} = 2$ one can use Proposition 10.2.1 (or rather its proof) to obtain explicit results.

In our actual example we have already determined all the primitive ideals of finite codimension in B^χ. Hence we also know all simple finite dimensional representations. It turns out that, for a fixed χ, they all lie in a unique $\mathcal{O}_\Gamma^{(1)}$. If χ is in family (B)(C) or (D) then for this Γ one has $|T^c| = 1$ (notation : §10). Then as in the proof of Proposition 10.2.1 one sees that the category of finite dimensional B^χ-modules is semi-simple.

Assume now that χ is in family (E). Following again the strategy of the proof of Proposition 10.2.1 we find that Q_f^p is given by the union of

$$\begin{tikzcd} \overset{\psi_2}{\bullet} \arrow[r, shift right] & \overset{\psi_4}{\bullet} \arrow[l, shift right] \arrow[r, shift right] & \overset{\psi_6}{\bullet} \arrow[l, shift right] \\ (\mathrm{oooo}) & (\mathrm{ooo\text{-}}) & (\mathrm{oo\text{-}\text{-}}) \end{tikzcd}$$

and

$$\begin{tikzcd} \overset{\psi_8}{\bullet} \arrow[r, shift right] & \overset{\psi_{10}}{\bullet} \arrow[l, shift right] \arrow[r, shift right] & \overset{\psi_{12}}{\bullet} \arrow[l, shift right] \\ (\text{-}\text{-}\text{-}\text{-}) & (\text{-}\text{-}\text{-}\mathrm{o}) & (\text{-}\text{-}\mathrm{oo}) \end{tikzcd}$$

In the indexation of the vertices we have written "-" for -1 and "o" for 0. We have also indicated the corresponding ψ's (see (10.2)). Invoking Theorem 10.1.4 we find that in this case the quiver describing the finite dimensional representations of B^χ is determined by the lattice of primitive ideals of B^χ. The correspondence is given in Table 11.2. We see that we have a semi-simple category unless we are in case 2. In that case B^χ has two simple finite dimensional representations, say L_1, L_2 and two non-simple indecomposable modules which are respectively an extension of L_1 by L_2 and an extension of L_2 by L_1.

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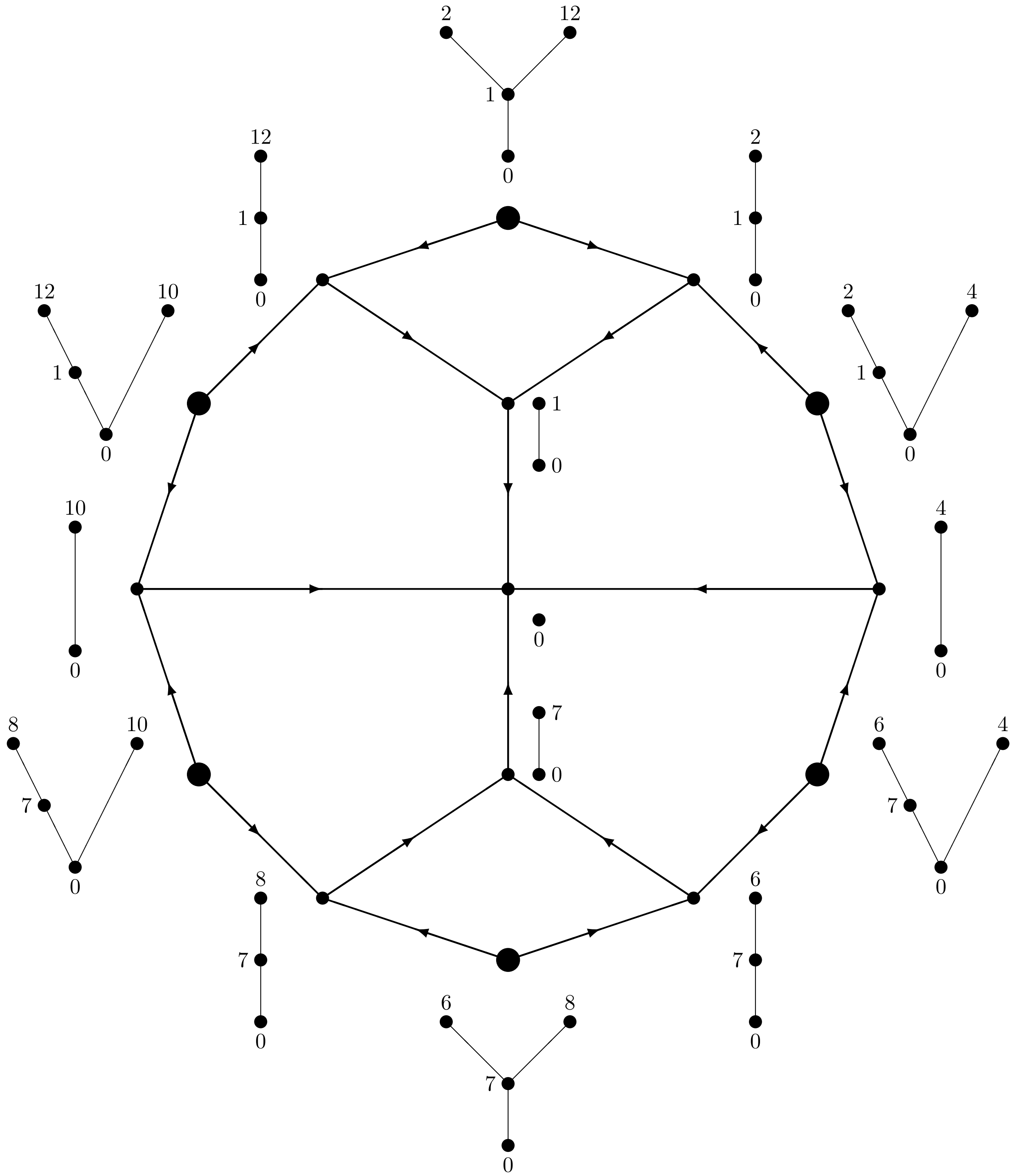

FIGURE 11.3. The lattices of primitive ideals of the B^χ's together with the $\to$-relation

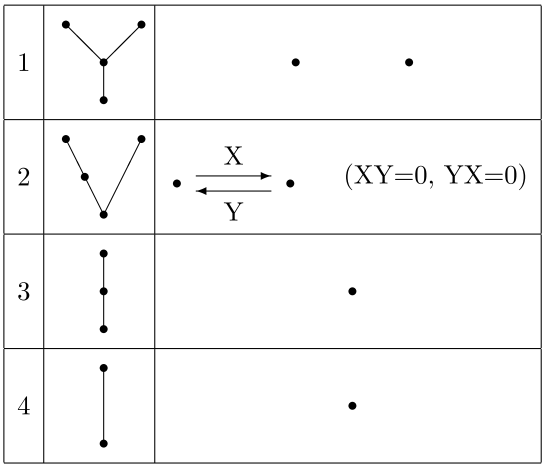

TABLE 11.2. The quivers determining the finite dimensional representations of B^χ

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